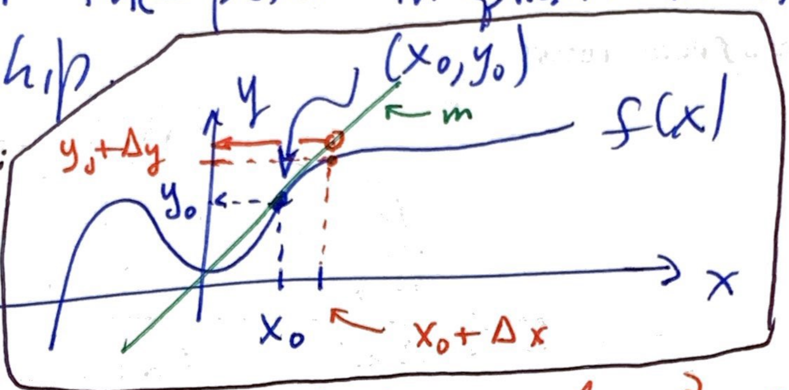
2.9 Differentials, Incrementals and Linear Approximations

I Linear approximation

- Sometimes manipulating an implicit relation is tedius, so we can use a linear approximation near the point in question of the relationship.
- The Set-Up:

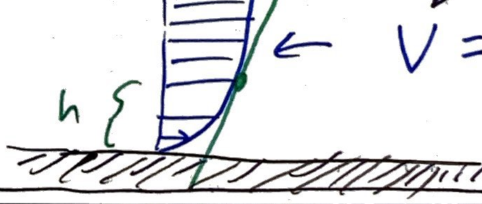


→ Sometimes its hard to eval. a relationship @ $x_0+\Delta x$ to get $y_0+\Delta y$

→ So we use a linear approximation via the tangent line

- We estimate Δy based on the tangent line vs. the actual curve.

Applied! Computational Fluid Dynamics.



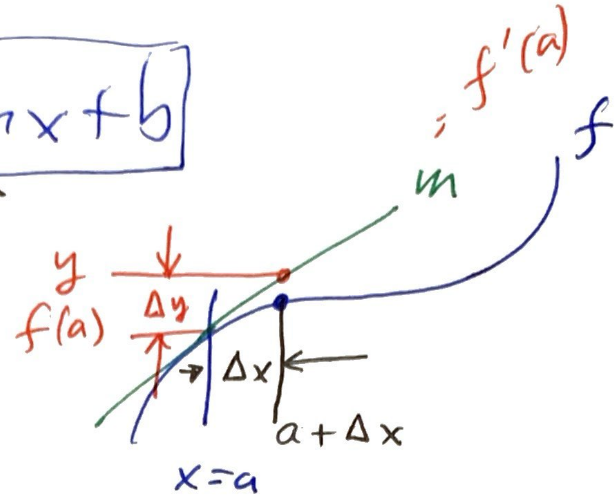
computationally expensive!

$$V = \frac{\sin^2(\sqrt{y}+a)}{y^{3/2}} \approx \underbrace{m \cdot y}_{\text{less computation}}$$

$\left.\frac{dv}{dy}\right|_{@\, y=h}$

• **Tangent Lines:** $\boxed{y = mx + b}$

exact slope of curve $\Big\}$ $m = \left.\dfrac{df}{dx}\right|_{x=a}$



Use as the slope of a line $\Big\}$ $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y - \overset{f(a)}{y_0}}{x - \underset{a}{x_0}}$

$\left.\dfrac{df(x)}{dx}\right|_{x=a}$

Together

$$\boxed{\frac{y - f(a)}{x - a} = f'(a)}$$

• We may use $\Delta y = y - f(a)$; $\Delta x = x - a$

$$\boxed{\frac{\Delta y}{\Delta x} = f'(a)}$$

incrementals

• The tangent line

$$\boxed{f(x) \approx f'(a)(x-a) + f(a)}$$

$$y \approx m(x-a) + y_a$$

Warning: use near $x = a$. Not acc't too far away from "a"

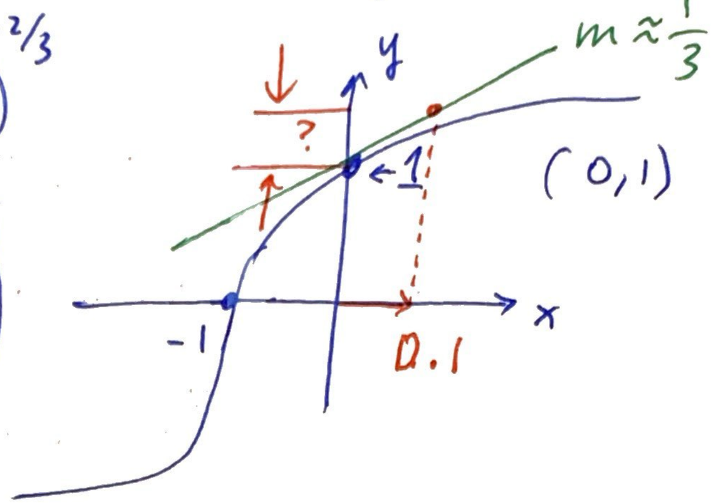
Ex let $f(x) = \sqrt[3]{1+x}$

3

then $\left.\frac{df}{dx} = \frac{1}{3}(1+x)^{-2/3}\right|_{x=0}$ $(a=0)$

then $\frac{\Delta y}{\Delta x} \approx \frac{1}{3}(1+0)^{-2/3}$

$$\boxed{\Delta y \approx \frac{1}{3}\Delta x}$$



$$\Delta y \approx \frac{1}{3}(0.1) = \underline{\underline{0.033}}$$

So if we slide over from 0 to 0.1
then the curves change can be approx.
as going from $y=1$ to $y = 1 + 0.033 = \underline{\underline{1.033}}$

Compare to exact value
$$f(0.1) = \sqrt[3]{1+0.1} = \sqrt[3]{1.1} \approx \underline{\underline{1.032}}$$

• What is the linear approximation line?

(0,1)

$$y = mx+b$$
$$y = \frac{1}{3}x+b$$
$$1 = \frac{1}{3}\cdot 0+b$$
$$b=1$$

Final

$$\boxed{y = \frac{1}{3}x+1}$$

linear approx to $\sqrt[3]{1+x}$

good if we stay close to $x=0$.

Ex approx $\sqrt[3]{0.95}$

$\sqrt[3]{1+x}$ $x=-0.05$ near "0"

• we can use the same eqn. of the linear approx.

$$\boxed{y=\frac{1}{3}x+1}$$

$$=\frac{1}{3}(-0.05)+1$$

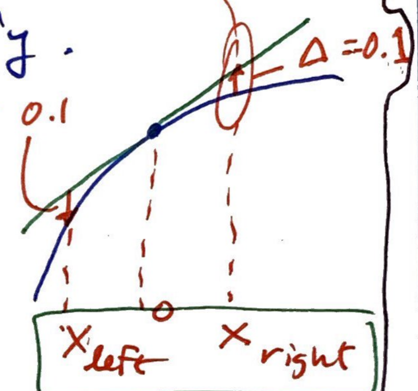
$= \underline{\underline{0.9833}}$ is the approx to $\sqrt[3]{0.95}$

BTW:
$$\boxed{\text{Exact } \sqrt[3]{0.95} = 0.9830}$$

0.0001

Q: For what regions of x will the linear approx. be within ± 0.1 acc'y.

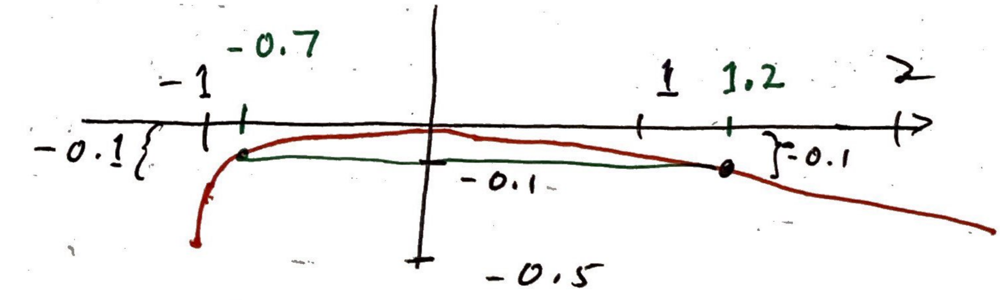
$$-0.1 < \underbrace{f}_{\text{curve}} - \underbrace{y_{tan}}_{\text{tangent line}} < \boxed{0.1}$$



Find these numbers

$$-0.1 < \underbrace{\sqrt[3]{1+x} - \left(\frac{1}{3}x+1\right)} < 0.1$$

Solve for x: desmos analysis



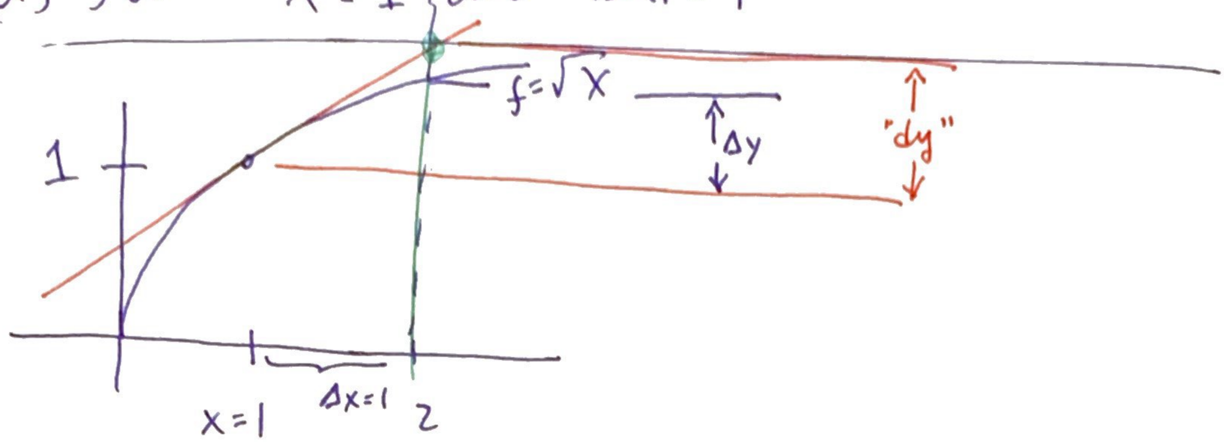
$$\boxed{\text{when } x \in [-0.7, 1.2]}$$

y_T is ± 0.1 acc

EX For $y=\sqrt{x}$ compute Δy and dy

(a) for $x=1$, and $\Delta x = 1$

(a)



$y = f(x) = \sqrt{x}$; curve $\begin{cases} \Delta y = f(2)-f(1) \\ \quad = \sqrt{2} - \sqrt{1} \\ \quad = \underline{\underline{0.41421}} \end{cases}$

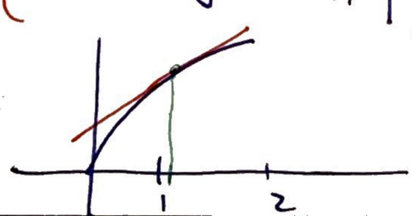
tangent line $\begin{cases} dy = y'dx \\ \quad = \frac{1}{2}x^{-1/2}\,dx \\ dy = \frac{1}{2\sqrt{x}}\,dx \Big|_{x=1} \\ dy = \frac{1}{2}\cdot 1 = \underline{\underline{0.5}} \end{cases}$

0.09 difference

(b) let $x=1$, $\Delta x = 0.01$ repeat above. We are much closer to 1

curve { • $\Delta y = f(1.01) - f(1) = \sqrt{1.01} - \sqrt{1} = \underline{\underline{0.00499}}$

tang. line { • $dy = \frac{1}{2\sqrt{x}}\Big|_{x=1} dx = \frac{1}{2\sqrt{1}}(0.01) = \underline{\underline{0.00500}}$



much closer difference of $\underline{\underline{0.0001}}$

Physics: The pendulum

PHYS 200

- Newton's Law $\boxed{\sum F = ma}$

Tangential Accel'n

tan: $-F_{grav}\sin\theta = m\,a_{Tan}$

$F_{grav} = mg$

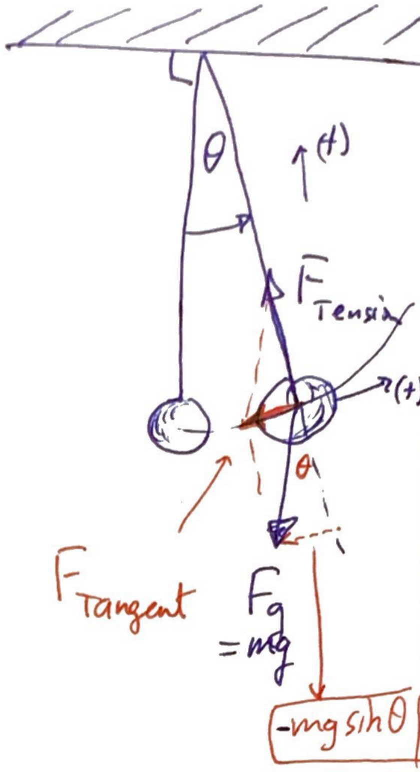
But convert linear to angular accl'n

linear acc'ln

ang. accl'n

$a_{Tan} = r\alpha$

$\alpha = \dfrac{d^2\theta}{dt^2}$, θ = angl'r displ.



$$-mg\sin\theta = mr\frac{d^2\theta}{dt^2}$$

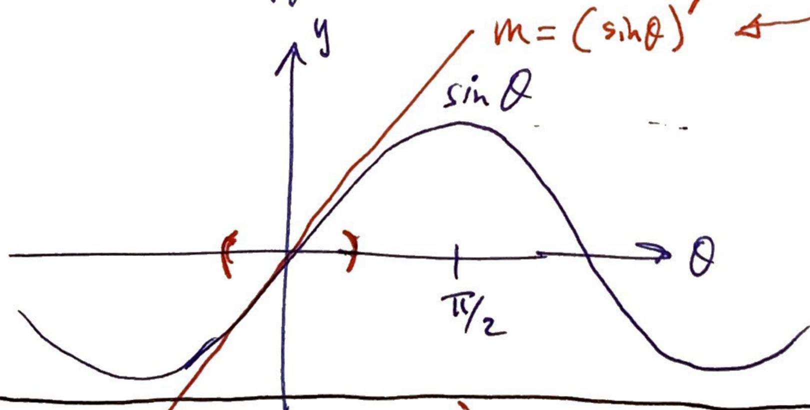
$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} + \frac{r}{g}\sin\theta = 0}$$

Differential equation $\theta = f(\text{time})$

need not remember this part.

↑ ↑ we can solve this but it is not pretty: ↑ ↑ ↑

- Lets approximate $\sin\theta$



Math 211

$f(\theta) \approx (\sin\theta)'(\theta - a) + f(a)$

$f(\theta) \approx \cos\theta\big|_{\theta=0}(\theta - 0) + f(0)$

$f(\theta) \approx 1 \cdot \theta$

Near $\theta = 0$

$\boxed{\sin\theta \approx \theta \text{ for small } \theta}$

Now lets approximate $\sin\theta \approx \theta$ in the Ordinary Differential Eqn (Good Near $\theta = 0$)

ODE: $\dfrac{d^2\theta}{dt^2} + \dfrac{r}{g}\theta = 0$

Soln: $\theta(t) = A\cos\left(\sqrt{\dfrac{r}{g}}\,t\right)$

- we noticed that $\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right) = 1$ in chpt 1.
- The ratio maintains itself as θ gets smaller and smaller.
- let $f(x) = \sin(x)$

$$\frac{df}{dx} = \cos(x)$$

finite changes

"infinitesimal changes"

then near $x = 0$ $\quad \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$

real world

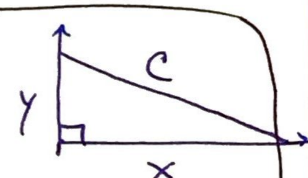
theory

yeilds

$$\Rightarrow \boxed{\Delta y \approx \frac{dy}{dx} \cdot \Delta x} \Rightarrow \Delta y \approx \underbrace{\cos \theta}_{\approx 1} \cdot \Delta \theta \Rightarrow \boxed{\sin \theta \approx \theta}$$

$(\theta \to 0)$

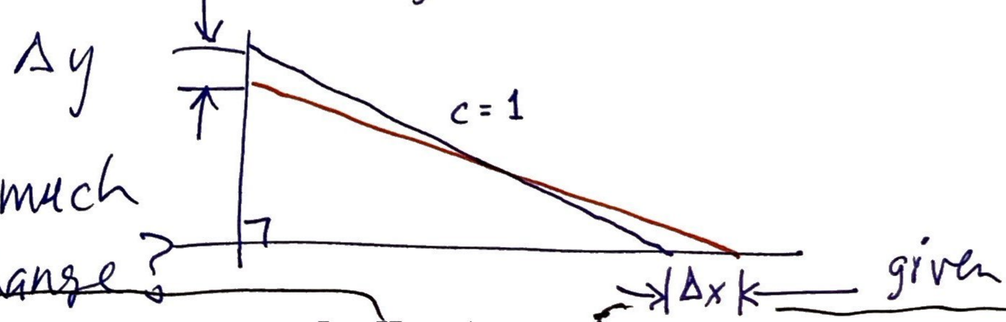
$\boxed{\text{Ex}}$ Pythagoras' Thm: $x^2 + y^2 = c^2$



lets Keep the hypotenueus fixed : $c \equiv 1$

then $x^2 + y^2 = 1$

- Now if we re-adjust the value of x a small amount, Δx, by how much does Δy change



Q: By how much will Δy change?

Ex (cont.)

- using differentials on the eqn:

$$x^2+y^2=1^2$$ we

implicitly differentiate

$$d(x^2+y^2=1^2)$$

$$dx^2+dy^2=d\cancel{1^2}^{\,0}$$

$$2x\,dx+2y\,dy=0$$ solve for dy

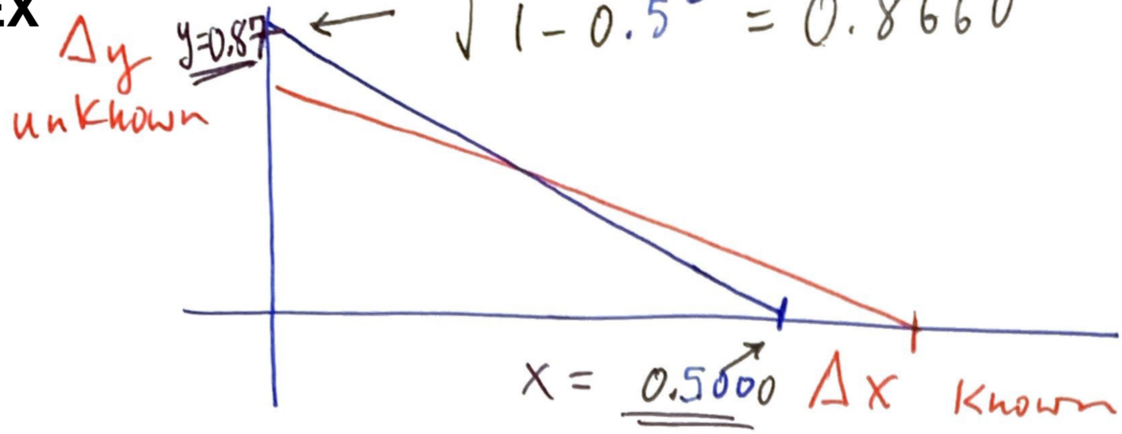
now $$dy=-\frac{x\,dx}{y}$$

$$\boxed{dy=-\left(\frac{x}{y}\right)dx}$$

The change in y as x changes

EX

$\sqrt{1-0.5^2} = 0.8660$



Q: Find Δy given a Δx change.

- Diff't $\quad x^2 + y^2 = 1^2$

$$2x\,dx + 2y\,dy = 0 \Rightarrow \boxed{dy = -\frac{x}{y}\,dx}$$

$x = 0.7$ we

change x by 0.1, say, [for $c=1$], then

y changes by $-\left(\frac{0.5000}{0.8660}\right)(0.1)$

$$\boxed{\Delta y = -0.0577}$$

- Then the new vertical location is $\approx$ @ $y = 0.8660 - 0.0577$

$$\boxed{y = 0.8083}$$

BTW:

- exact location of y:

$$x^2 + y^2 = 1^2$$

$$y = \sqrt{1-(0.500+0.1)^2}$$

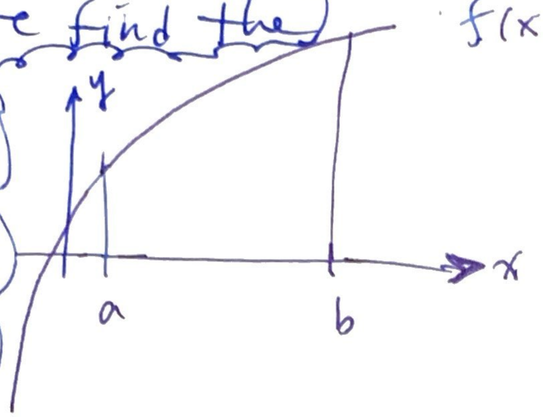
$$\underline{y = 0.8000}$$

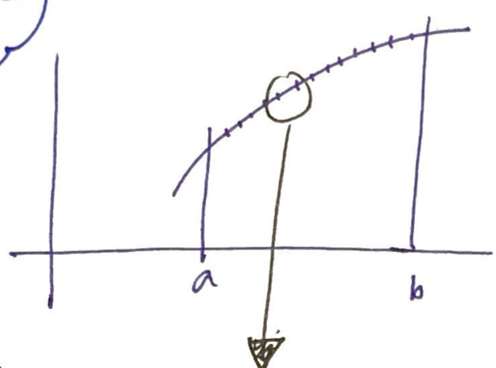
about 2 decimal place acc'y.

⊛ Differentials as variables (cont)

Ex length of a curve (Calc II)

- In Calc II we find the length of a curve by chopping it into lots of small pieces. Then we take the limit.



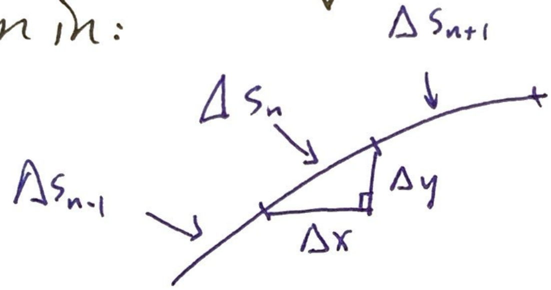


length of curve between $x=a$ to $x=b$ is

$$L \approx \sum_{n=1}^{N} \Delta s$$

sum up all segments

- Zoom in:



- Approximate length...

$$L \approx \sum_{n \to 1}^{N} \sqrt{\Delta x^2 + \Delta y^2}$$

- We take the limit as $\Delta x \to 0$. We convert this $\boxed{\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x) \cdot \Delta x}$ is written as $\boxed{\int f(x)\,dx}$. So, for arc length

So the exact answer

$$L = \int_{x=a}^{x=b} \underbrace{\sqrt{dx^2 + dy^2}}_{dS} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

factor out dx

> " $\int$ " is called an integral. You need not know or use it until chapter 4. I am just showing you how we use differentials