2.6 Implicit Differentiation

I We cannot always solve an equation for y so that we have $y = f(x)$.

Ex $x + y = (x^2 + y^2)^2$ ← not easy to solve for y

{if we convert this to polar coordinates then the problem is tractable]

- We can still discover the rate of change of y with respect to x by using a method called <u>implicit differentiation</u>

II Implicit Differentiation

ex

Consider $x+y^2=8$

- Take the derivative of both sides

$$\frac{d(x+y^2)}{dx}=\frac{d8}{dx}$$

$$\frac{dx}{dx}+\frac{dy^2}{dx}=0$$

$$1+2y\frac{dy}{dx}=0$$

solve for y'

$$\boxed{\frac{dy}{dx}=\frac{-1}{2y}}=\frac{\mp 1}{2\sqrt{8-x}}$$

{ but $y=\pm\sqrt{8-x}$

FYI Here we can solve for y

{ $y^2=8-x \Rightarrow y=\pm\sqrt{8-x}$

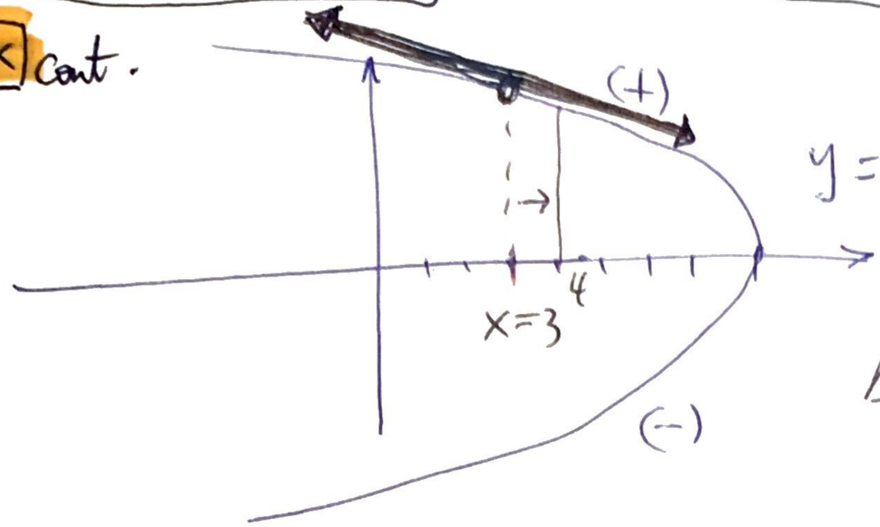
$$y'=\frac{\pm 1}{2}\cdot\frac{1}{\sqrt{8-x}}(8-x)'$$

$$y'=\frac{\mp 1}{2\sqrt{8-x}}$$ }

For the curve $x+y^2=8 \Rightarrow \frac{dy}{dx}=\frac{\mp 1}{2\sqrt{8-x}}$

- So if we approx. the tangent line's slope by finite changes vs. infinitesmal changes
- we can vary Δx and use the rate of change to calculate Δy ...

Ex cont.



$$y = \pm\sqrt{8-x}$$

$$\Delta x = 4-3 = 1$$

- If we move from 3 to 4 in x, how much, approximately, will Δy change?

$$\frac{\Delta y}{\Delta x} \approx \frac{\mp 1}{2\sqrt{8-x}}$$

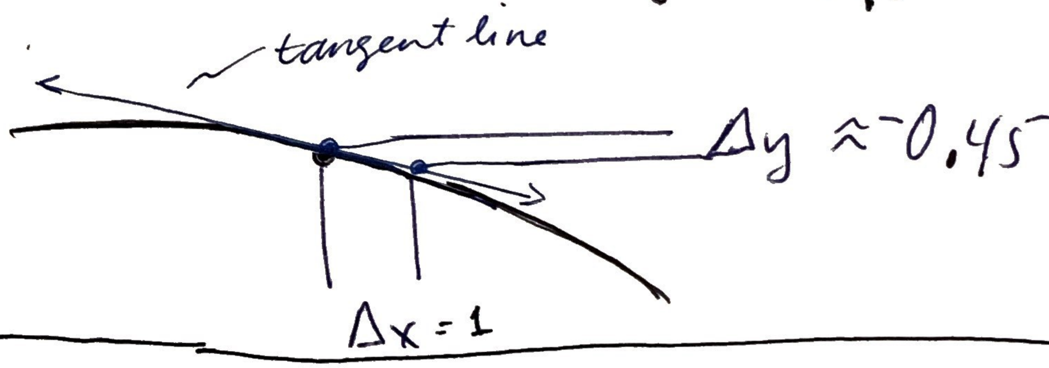
Top branch

@ $x=3$ $\quad \dfrac{\Delta y}{\Delta x} \approx \dfrac{\mp 1}{2\sqrt{8-3}}$ or $\dfrac{-1}{2\sqrt{5}}$

$$\boxed{\Delta y \approx \left(\frac{-1}{2\sqrt{5}}\right)\Delta x}$$

Ex

- If $\Delta x = 1$ we have $\Delta y \approx \dfrac{-1}{2\sqrt{5}}(1) \approx \underline{-0.45}$



Find the eqn of the tangent line to the curve $\boxed{y^2(y^2-4) = x^2(x^2-5)}$ at the point $(x,y) = (0,-2)$

Test: $(-2)^2\underbrace{((-2)^2-4)}_{0} \stackrel{?}{=} 0^2 \cdot (0^2-5)$?

$0 = 0$ yes

Differentiate both sides of the eqn:

1st multiply the y^2 through on the LHS. ditto on R

$$\underline{\underline{y^4 - 4y^2 = x^4 - 5x^2}}$$

{to avoid product

• Now Diff't both sides

$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3\cancel{\frac{dx}{dx}}^{1} - 5\cdot 2x\cancel{\frac{dx}{dx}}$$

• Solve for $\frac{dy}{dx}$: $\frac{dy}{dx}\left[4y^3-8y\right] = 4x^3-10x$

$$\boxed{\frac{dy}{dx} = \frac{4x^3-10x}{4y^3-8y}}$$

This is an implicit function: $dy/dx = F(x,y)$

BTW: 2.7 functions are $= f(x$

EX cont. Now evaluate the slope at the given pt: $\left.\frac{dy}{dx}\right|_{(0,-2)} = \left.\left(\frac{4x^3-10x}{4y^3-8y}\right)\right|_{(0,-2)}$ x y

$$= \frac{4\cdot 0^3 - 10\cdot 0}{4(-2)^3 - 8(-2)}$$

$$= \frac{0}{-32+16}$$

$$= \frac{0}{-16}$$

$$= \boxed{0}$$

- eqn of tangent line at $(0,-2)$ is

Form $y = mx + b$

Point $y = (0)x + b$

Solve $-2 = (0)(0) + b \rightarrow b = -2$

Final $y = 0x - 2$

$$\boxed{y = -2}$$

- Go to desmos.com and enter the implicit eqn and check the tangent at $x=0, y=-2$

EX

Use implicit diff'n on $x^2 + xy = 10$ to find $y'(2)$

(i) @ $x = 2$ what is y? $(x,y) = (2, \underline{?})$

$2^2 + 2 \cdot y = 10 \rightarrow \boxed{y=3}$ $= \boxed{(2,3)}$

(ii) Implicit diff'n $\rightarrow$ solve for $\frac{dy}{dx}$

$2x + (x'y + xy') = 10 \rightarrow 2x + 1 \cdot y + x \cdot y' = 0$

(iii) plug in (x,y) @ point $(2,3)$:

$y' = \frac{-2x - y}{x}$

$\boxed{y' = -2 - y/x}$

$y'\Big|_{(2,3)} = -2 - \left(\frac{3}{2}\right) = -\frac{4}{2} - \frac{3}{2} = \boxed{-\frac{7}{2}}$

(iv) Form the eqn of tan line:

$y = mx + b$

Form $y = -\frac{7}{2}x + b$

point $3 = -\frac{7}{2}(2) + b$

solve $3 = -7 + b \rightarrow b = 10$

Fnl: $\boxed{y = -\frac{7}{2}x + 10}$