2.5 The Chain Rule

This section addresses the derivative of a composite function, $f \circ g$

I The chain rule

The Chain Rule

If g is diff'ble @ x and
f is diff'ble @ $g(x)$
then $f(g(x))$ is differentiable at x and

$$\boxed{\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)}$$

EX Find $f'(x)$ if $f(x) = \cos(\sin(x))$

$$\left(\cos(\sin(x))\right)' = -\sin(\sin(x)) \cdot \frac{d \sin(x)}{dx}$$

$$= \boxed{-\sin(\sin(x)) \cdot \cos(x)}$$

(*) Chain Rule with the power function:

$$\boxed{\frac{d\,(u^n)}{dx} = n u^{n-1} \cdot \frac{du}{dx}}$$

in liebniz notation:

$$\boxed{\frac{d\, g^n(x)}{dx} = n\, g^{n-1}(x) \cdot \frac{d\, g(x)}{dx}}$$

II Chain Rule applied to basic rules

- $\frac{dx^n}{dx} = n x^{n-1}$ but if we replace x with $u(x)$ then the rule becomes

$$\boxed{\frac{du^n}{dx} = n u^{n-1} \cdot u'(x)}$$

EX Differentiate $y = (x^3+3x^2+1)^{3/2}$

$$\left[(x^3+3x^2+1)^{3/2}\right]' = \frac{3}{2}(x^3+3x^2+1)^{3/2-1} \cdot (x^3+3x^2+1)'$$

$$= \boxed{\frac{3}{2}(x^3+3x^2+1)^{1/2} \cdot (3x^2+6x)}$$

Ex Find $\left(\sin(x^2)\right)'$

let $u = x^2$

So $\left(\sin(x^2)\right)' = [\sin(x)]'\Big|_{x=u} \cdot u'$

$= \cos(x)\Big|_{x=x^2} \cdot (x^2)'$

$= \boxed{\cos(x^2)\cdot 2x}$

Ex Find $[\tan(\sin(x))]'$

$= \sec^2(\sin(x)) \cdot [\sin(x)]'$

$= \boxed{\sec^2(\sin(x))\cos(x)}$

Ex

Find $\frac{d}{dx}\left[\underbrace{\sin^2(x)}_{u}\cdot\underbrace{\cos(x)}_{v}\right]$

$$= \frac{d(uv)}{dx}$$

$$= \left(\frac{du}{dx}\right)v + u\left(\frac{dv}{dx}\right)$$

$$= \left(\frac{d\sin^2(x)}{dx}\right)\cdot\cos(x) + \sin^2(x)\left(\frac{d\cos(x)}{dx}\right)$$

$$= \left(\frac{d[\sin(x)]^2}{dx}\right)\cdot\cos(x) + \sin^2(x)[-\sin(x)]$$

$$= 2[\sin(x)]^{2-1}\cdot\frac{d\sin(x)}{dx}\cdot\cos(x) - \sin^3(x)$$

$$= 2\sin(x)\cdot\cos(x)\cdot\cos(x) - \sin^3(x)$$

$$= \boxed{2\sin(x)\cos^2(x) - \sin^3(x)}$$

Ex

$$\left(\frac{(x+1)^3}{(x^2-1)^2}\right)'$$

let $u = (x+1)^3$

$V = (x^2-1)^2$

So $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$$= \frac{[(x+1)^3]'(x^2-1)^2 - (x+1)^3[(x^2-1)^2]'}{(x^2-1)^4}$$

- $[(x+1)^3]'$: let $w = x+1$

then $[w^3]'$

$= 3w^2 \cdot w' = 3(x+1)^2(x+1)'$

$= 3(x+1)^2 \cdot (1+0) = \underline{3(x+1)^2}$

- $[(x^2-1)^2]'$ let $s = x^2-1$

then $[s^2]' = 2s \cdot s'$

$= 2(x^2-1)(x^2-1)'$

$= 2(x^2-1)(2x-0)$

$= 2(x^2-1)(2x) = 4x(x^2-1)$

Ex (cont) ⑥

piece the quotient rule together

$$\left(\frac{u}{v}\right)' = \boxed{\frac{[3(x+1)^2](x^2-1)^2 - (x+1)^3 4x(x^2-1)}{(x^2-1)^4}}$$

$$= \frac{(x+1)^2 \cancel{(x^2-1)}[3(x^2-1) - (x+1)4x]}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{(x+1)^2 [3x^2-3-4x^2-4x]}{(x^2-1)^3}$$

$$= \frac{(x+1)^2 [-x^2-4x-3]}{(x^2-1)^3}$$

$$= \boxed{\frac{-(x+1)^3(x+3)}{(x^2-1)^3}}$$

Ex Find y' if $y = \dfrac{(x+1)^3}{(x^2-1)^2}$, again w/o u&v this time...

$$\boxed{\text{Quotient rule}: \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

So $$\left[\frac{(x+1)^3}{(x^2-1)^2}\right]' = \frac{\left[(x+1)^3\right]'(x^2-1)^2 - (x+1)^3 \cdot \left[(x^2-1)^2\right]'}{\left[(x^2-1)^2\right]^2}$$

$$= \frac{3(x+1)^2 \cdot (x+1)' (x^2-1)^2 - (x+1)^3 2(x^2-1)(x^2-1)'}{(x^2-1)^4}$$

$$= \frac{3(x+1)^2 \cdot 1 (x^2-1)^2 - (x+1)^3 2(x^2-1) \cdot 2x}{(x^2-1)^4}$$

$$= \frac{\cancel{(x^2-1)}(x+1)^2\left[3(x^2-1) - 4(x+1)x\right]}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{(x+1)^2\left[3x^2 - 3 - 4x^2 - 4x\right]}{(x^2-1)^3}$$

$$= \frac{(x+1)^2\left[-x^2 - 4x - 3\right]}{(x^2-1)^3}$$

$$= \frac{-(x+1)^2(x+3)(x+1)}{(x^2-1)^3} = \boxed{-\frac{(x+1)^3(x+3)}{(x^2-1)^3}}$$

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Utilize the previous problems in the hierarchy to answer each:

(1) If $f(x) = \sin(3x)$ Find $f'(x)$

$f' = (\sin(3x))' = \cos(3x) \cdot (3x)' = \boxed{3\cos(3x)}$

(2) $g(x) = (\sin(3x))^3$ Find $g'(x)$

$g' = 3(\sin(3x))^2 \cdot (\sin(3x))'$

$g' = 3\sin^2(3x) \cdot 3\cos(3x)$

$\boxed{g' = 9\sin^2(3x)\cos(3x)}$

$(\sin x)^2 = \sin^2(x)$

(3) $h(x) = (\sin(3x))^3 + 5x$ $h'(x) = ?$

$h'(x) = (\sin^3(3x) + 5x)'$ sum rule

$= (\sin^3(3x))' + (5x)'$

$\boxed{h' = 9\sin^2(3x)\cos(3x) + 5}$

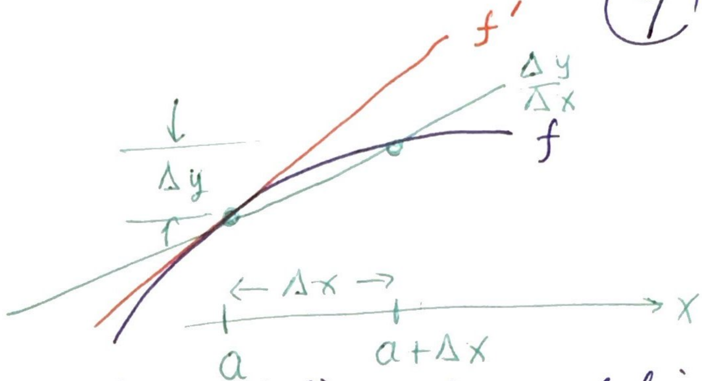
(4) $j(x) = [(\sin(3x))^3 + 5x]^5$ find $j'(x)$

$j'(x) = \left([(\sin(3x))^3 + 5x]^5\right)'$

$= 5[\quad]^4 \cdot [\quad]'$

$\boxed{j' = 5[\sin^3(3x) + 5x]^4 \cdot (9\sin^2(3x)\cos(3x) + 5)}$

* "proof" of Chain Rule



(i) preliminaries

$$\Delta y = f(a+\Delta x) - f(a)$$

define

$$\boxed{\varepsilon \equiv \frac{\Delta y}{\Delta x} - f'(a)}$$ difference of slopes between tangent line and secant line

• Note: $\lim\limits_{\Delta x \to 0} \varepsilon = \lim\limits_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a)\right) = f'(a) - f'(a) = \underline{\underline{0}}$

• Solve for Δy: $\boxed{\Delta y = f'(a)\Delta x + \varepsilon \Delta x}$

(ii) Consider f composited with g: $f(g(x))$

let $u = g(x)$, diff'ble @ a

then $y = f(u)$, diff'ble @ $\underline{b \text{ which is } g(a)}$

Next, use Δy formula from (i) <u>twice</u>:

$$\Delta u = g'(a) \cdot \Delta x + \varepsilon_1 \Delta x$$

$$\Delta y = f'(b) \cdot \Delta u + \varepsilon_2 \Delta u$$
$$= [f'(b) + \varepsilon_1]\Delta u$$

(iii) <u>substitute Δu into Δy</u>

$$\Delta y = [f'(b) + \varepsilon_2][g'(a)\Delta x + \varepsilon_1 \Delta x]$$

$\div \Delta x$ $$\frac{\Delta y}{\Delta x} = (f'(b) + \varepsilon_2)(g'(a) + \varepsilon_1)$$

(iv) take the limit:

$$\lim \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (f'(b) + \cancel{\varepsilon_2}^{\,0})(g'(a) + \cancel{\varepsilon_1}^{\,0})$$

$$y' = f'(b) \cdot g'(a)$$ but $b = g(a)$

$$\Longrightarrow \boxed{y' = f'(g(a)) \cdot g'(a)}$$

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