# 2.4 Derivatives of Trig Functions



## I $(\sin(x))'$

- Def $$\frac{d\sin(x)}{dx} \equiv \lim_{h\to 0}\left(\frac{\sin(x+h) - \sin(x)}{h}\right)$$

- Identity use $$\boxed{\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)}$$

- So $$\frac{d\sin(x)}{dx} = \lim_{h\to 0}\left[\frac{[\sin(x)\cos(h) + \cos(x)\sin(h)] - \sin(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)\cos(h) - \sin(x)}{h} + \cos(x)\frac{\sin(h)}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin(x)\left[\frac{\cos(h)-1}{h}\right]\right] + \lim_{h\to 0}\left[\cos(x)\frac{\sin(h)}{h}\right]$$

$$= \sin(x)\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right] + \cos(x)\lim_{h\to 0}\left[\cancel{\frac{\sin(h)}{h}}^{\,1}\right]$$

| $h$ | $\frac{\cos(h)-1}{h}$ |
|---|---|
| 0.1 | $[\cos(0.1)-1]/0.1 = -0.04$ |
| 0.01 | $[\cos(0.01)-1]/0.01 = -0.004$ |
| 0.001 | $[\cos(0.001)-1]/0.001 = -0.00005$ |
| -0.001 | $-[\cos(-0.001)-1]/0.001 = +0.00005$ |
| -0.01 | $-[\cos(-0.01)-1]/0.01 = 0.004$ |
| -0.1 | $-[\cos(-0.1)-1]/0.1 = -0.04$ |

So $\lim_{h\to 0}\frac{\cos(h)-1}{h} = \boxed{0}$

$$= \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$\boxed{\frac{d\sin(x)}{dx} = \cos(x)}$$

**Ex** Find $\lim_{x \to 0} \left[ \frac{\sin(2x)\sin(6x)}{x^2} \right]$

- "Trick" $\lim_{x \to 0} \left[ \frac{\sin(2x)}{x} \cdot \frac{\sin(6x)}{x} \right]$

Substitution: let $u = 2x \nearrow 0$ and $v = 6x \nearrow 0$

$$\lim_{x \to 0} \left[ \frac{\sin(u)}{u/2} \cdot \frac{\sin(v)}{v/6} \right]$$

$$= 2 \cdot 6 \lim_{u \to 0} \left[ \frac{\sin(u)}{u} \right] \cdot \lim_{v \to 0} \left[ \frac{\sin(v)}{v} \right]$$

$$= 12 \cdot 1 \cdot 1 = \boxed{12}$$

## II $(\cos(x))'$

We can use similar identities and show that

$$\boxed{\frac{d\cos(x)}{dx} = -\sin(x)}$$

III $(\tan(x))'$



$$\frac{d\tan(x)}{dx}$$

$$= \frac{d\left(\frac{\sin(x)}{\cos(x)}\right)}{dx}$$

$$= \frac{(\sin(x))'\cos(x) - \sin(x)(\cos(x))'}{\cos^2 x}$$

$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2 x}$$

$$= \frac{c^2x + s^2x}{\cos^2 x}$$

$$= \frac{1}{\cos^2(x)}$$

$$= \sec^2(x)$$

$$\boxed{\frac{d\tan(x)}{dx} = \sec^2(x)}$$

# Summary of trig derivatives

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\csc(x)) = -\cos(x)\cot(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x) \qquad \frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dt}(\tan(x)) = \sec^2(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

**Ex** diff't $f(x) = \dfrac{1-\sec(x)}{\tan(x)}$

$$\left(\frac{1-\sec(x)}{\tan(x)}\right)' = \frac{(1-\sec(x))'\tan(x) - (1-\sec(x))(\tan(x))'}{\tan^2(x)}$$

$$= \frac{(0-\sec(x)\tan(x))\tan(x) - (1-\sec(x))\sec^2(x)}{\tan^2(x)}$$

$$= \frac{-\sec(x)\tan^2(x) - \sec^2(x) + \sec^3(x)}{\tan^2(x)}$$

$$= \frac{-\sec(x)\underbrace{[\tan^2(x) - \sec^2(x)]}_{-1} - \sec^2(x)}{\tan^2(x)}$$

$$= \boxed{\frac{\sec(x) - \sec^2(x)}{\tan^2(x)}}$$

$c^2 + s^2 = 1 \quad \div c^2$

$1 + \frac{s^2}{c^2} = \frac{1}{c^2}$

$1 + \tan^2 = \sec^2$

$\Rightarrow \boxed{\tan^2 - \sec^2 = -1}$