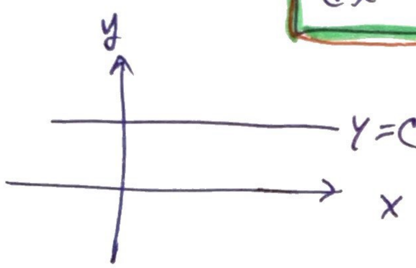
2.3 Differentiation Formulas

I Consider $f(x) = c$ $\quad c$ = const value

then $f' = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{c-c}{h} = \boxed{0}$

So $\boxed{\dfrac{d}{dx}(c) = 0}$, $\dfrac{df}{dx} = 0$ if $f = c$, $f' = 0$



$y = c$, slope $= 0$ so $f' = 0$

II Power functions: $f(x) = x^n$, n is pos. integer

1st recall Binomial expansion:

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

- Now $f'(x) = \lim_{h\to 0} \dfrac{(x+h)^n - x^n}{h}$

$$f' = \lim_{h\to 0} \frac{\left[\cancel{x^n} + nx^{n-1}h + \frac{(n)(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - \cancel{x^n}}{h}$$

$$f' = \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right)$$

$$f' = nx^{n-1}$$

$\boxed{\text{So If } f(x) = x^n \text{ then } \boxed{f'(x) = nx^{n-1}}}$ n needs to be (+) integer

III Constant multiple rule

let $g(x) = c f(x)$ where c = constant.

Then

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = cf'$$

If $\boxed{g = cf \text{ then } \boxed{g' = cf'}}$

IV Sum Rule

$$\frac{d(f+g)}{dx} = \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f' + g'$$

Likewise for $f - g$.

Summary

$$(f+g)' = f' + g'$$
$$(f-g)' = f' - g'$$

$$\boxed{(f \pm g)' = f' \pm g'}$$

EX Find the eqn of the tangent line to $y = 3x^4 + 2x^2 + 1$ @ $(1,6)$

- line: $y = mx + b$ we need "m" at any x value
- Slope: $m = f'(x)$

$$\frac{dy}{dx} = \frac{d\,(3x^4 + 2x^2 + 1)}{dx}$$

sum rule

$$= \frac{d\,3x^4}{dx} + \frac{d\,2x^2}{dx} + \frac{d\,1}{dx}$$

Const. mult. rule. ⊕ deriv. of const

$$= 3\,\frac{dx^4}{dx} + 2\,\frac{dx^2}{dx} + 0$$

power rule

$$= 3 \cdot 4x^{4-1} + 2 \cdot 2x^{2-1}$$

$$\boxed{y' = 12x^3 + 4x}$$

So $m\Big|_{(1,6)} = (12x^3 + 4x)\Big|_{x=1} = 12 + 4 = \underline{\underline{16}}$

- Equation:

Form: $y = 16x + b$

Point: $6 = 16 \cdot 1 + b$

Solve: $6 - 16 = b \rightarrow b = -10$

Final: $\boxed{y = 16x - 10}$

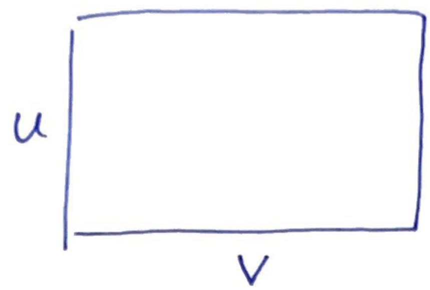
↳ FPSF "spit method"

armani method

Lin. Algebra $[A \mid I]$ is called the

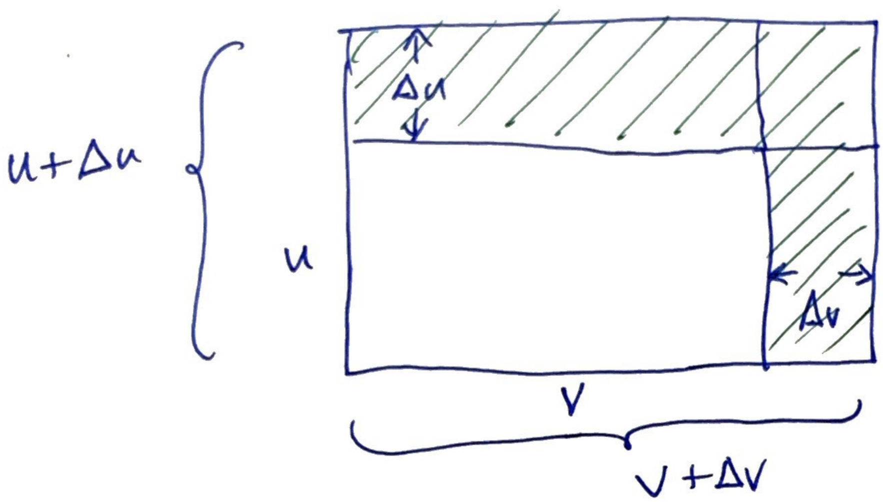
V Product Rule

Consider the area of a rectangle



Area $= u \cdot v$

Increase u and v by small amounts



$$\text{New Area} = (u+\Delta u)(v+\Delta v)$$
$$= uv + v\Delta u + u\Delta v + \Delta u \cdot \Delta v$$

$$\Delta \text{Area} = \text{Area}_{new} - \text{Area}_{old}$$
$$= [\cancel{u \cdot v} + v\Delta u + u\Delta v + \Delta u \Delta v] - [\cancel{u \cdot v}]$$

$$\boxed{\Delta A = v\,\Delta u + u\,\Delta v + \Delta u \Delta v}$$

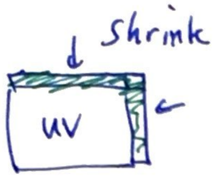
$\div \Delta x$
$$\frac{\Delta A}{\Delta x} = v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \frac{\Delta u \Delta v}{\Delta x}$$

delta's.
Finite Changes

Now take the limit as $\Delta x \to 0$

differentials

Infinitesimal changes



$$\frac{dA}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + \frac{du \cdot dv}{dx}$$

$$\frac{d(u \cdot v)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + \underline{\mathcal{O}(d\ell)^2}$$

order of magnitude smaller

let $u = f(x)$, let $v = g(x)$

then $$\boxed{\frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx}\cdot g}$$ product rule

Note: $$\boxed{\frac{d(fgh)}{dx} = f'gh + fg'h + fgh'}$$

Ex diff't $f(x) = (x^2 - 4x^4)(x + 5x^3)$

$$f' = (x^2-4x^4)' \cdot (x+5x^3) + (x^2-4x^4)\cdot(x+5x^3)'$$

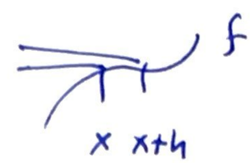
$$= \left[(x^2)' - 4(x^4)'\right](x+5x^3) + (x^2-4x^4)\left[x' + 5(x^3)'\right]$$

$$= (2x - 16x^3)(x+5x^3) + (x^2-4x^4)(1+15x^2)$$

$$= 2x^2 - 16x^4 + 10x^4 - 80x^6 + x^2 - 4x^4 + 15x^4 - 60x^6$$

$$\boxed{f'(x) = -140x^6 + 5x^4 + 3x^2}$$

Quotient Rule



- let $u = f(x)$, $v = g(x)$

then $\begin{cases} \Delta u = f(x+\Delta x) - f(x) \\ \Delta v = g(x+\Delta x) - g(x) \end{cases}$

- ratio

the change of the ration $\left(\frac{u}{v}\right)$ can be written as

$$\Delta\left(\frac{u}{v}\right) \equiv \left(\frac{u+\Delta u}{v+\Delta v}\right) - \left(\frac{u}{v}\right)$$

- common denominator

$$\Delta\left(\frac{u}{v}\right) = \frac{(u+\Delta u)v - u(v+\Delta v)}{(v+\Delta v)(v)}$$

$$= \frac{v\Delta u + \cancel{uv} - \cancel{uv} - u\Delta v}{v^2 + v\Delta v}$$

$$\left\{ \Delta\left(\frac{u}{v}\right) = \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v} \right\} / \Delta x$$

$$\frac{\Delta(u/v)}{\Delta x} = \frac{v\,\Delta u/\Delta x - u\,\Delta v/\Delta x}{v^2 + v\Delta v}$$

- Limit $\Delta x \to 0$

$$\boxed{\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}} \quad \boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

$$\left(\frac{\square}{\Delta}\right)' = \frac{\square'\Delta - \square\Delta'}{\Delta^2}$$

EX If $f(x) = \dfrac{x^2}{1+2x}$ Find f'

$$f' = \frac{(x^2)'(1+2x) - (x^2)(1+2x)'}{(1+2x)^2}$$

$$f' = \frac{2x(1+2x) - x^2(2)}{(1+2x)^2}$$

$$f' = \frac{4x^2+2x-2x^2}{(1+2x)^2} = \frac{2x^2+2x}{(1+2x)^2}$$

$$\boxed{f' = \frac{2x(1+x)}{(1+2x)^2}}$$

General Power Rule

In a future lecture (Calc II)

the power rule holds for all real numbers not just integers

$$\boxed{\frac{d x^n}{dx} = n x^{n-1}}$$

for all $n \in \mathbb{R}$

element of

real #'s

EX let $g(x) = x^{3/2}$, Find g'

$$\frac{d x^{3/2}}{dx} = \frac{3}{2} x^{3/2 - 1}$$

$$= \boxed{\frac{3}{2} x^{1/2}}$$

EX let $h(x) = x^{-7.2}$, Find h'

$$\frac{d x^{-7.2}}{dx}$$

$$= -7.2\, x^{-7.2-1}$$

$$= \boxed{-7.2\, x^{-8.2}}$$

Application

Find the tangent line to the curve

$$y = x^{4/3} \ @ \ (8,16)$$

- Test $8^{4/3} = \left(8^{1/3}\right)^4 = (2)^4 = 16$

- $m = \dfrac{d x^{4/3}}{dx}$

$$= \frac{4}{3} x^{\frac{4}{3}-1}$$

$$m = \frac{4}{3} x^{1/3} \ @ \ x = 8 \Rightarrow m = \frac{4}{3}\sqrt[3]{8}$$

$$\boxed{m = \frac{8}{3}}$$

- line

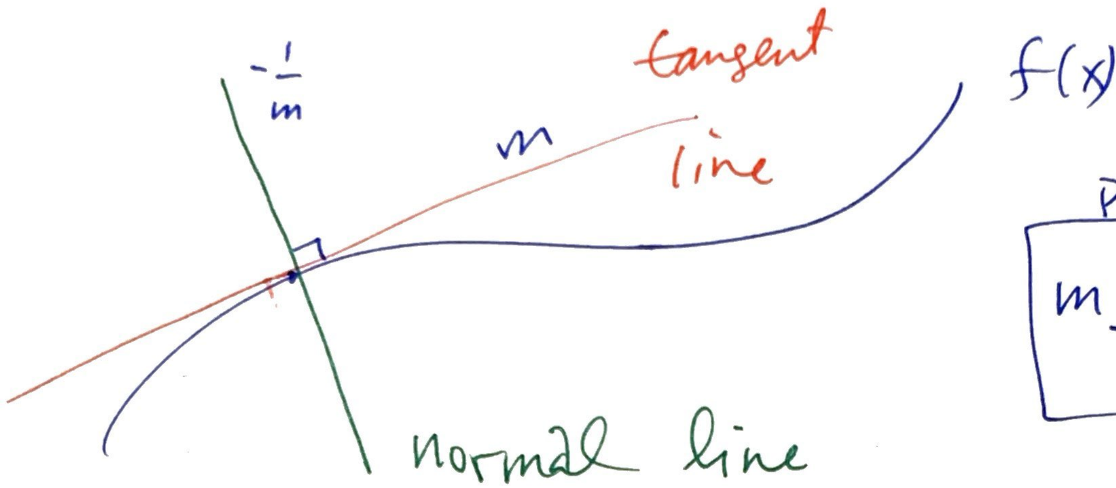
$y = \frac{8}{3} x + b$ form

$16 = \frac{8}{3}(8) + b$ point

$\left.\begin{aligned} b &= 16 - \frac{64}{3} \\ b &= \frac{48-64}{3} \\ b &= -\frac{16}{3} \end{aligned}\right\}$ solve

$\boxed{y = \frac{8}{3}x - \frac{16}{3}}$ tangent line

Normal Line



perp. lines

$$\boxed{m_\perp = \frac{-1}{m}}$$

IF $y = mx+b$ is the tangent line

then $y = \frac{-1}{m}x + d$ is the normal line

EX Previous example

$$y_{tan} = \frac{8}{3}x - \frac{16}{3} \text{ @ } (8,16)$$

$$y_{normal} = \left(-\frac{3}{8}\right)x + d \quad \text{form}$$

$$16 = -\frac{3}{\cancel{8}} \cdot \cancel{8} + d \rightarrow d = 16 + 3 = 19$$

$$\boxed{y_{normal} = \frac{-3}{8}x + 19}$$