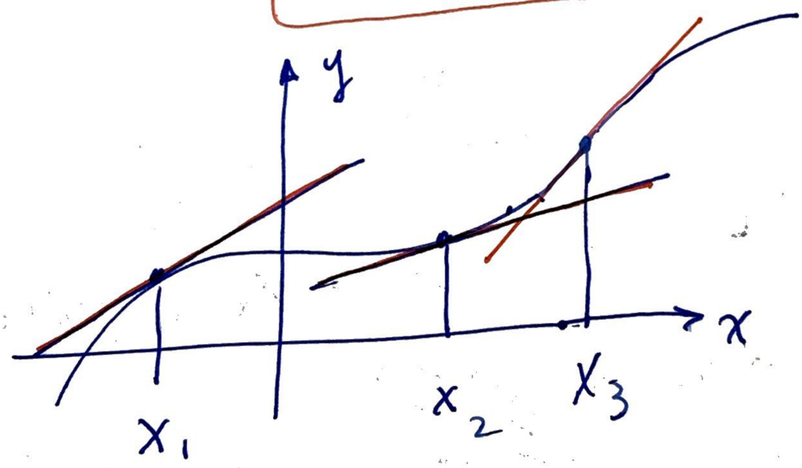
2.2 Derivative functions

I The derivative as a function of x

- To date we have been evaluating the slope of a function at a fixed point.
- We now let that fixed point "float" on the curve
  - fixed $f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$
  - float $\boxed{f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]}$



m = function of x

Ex Find $f'(x)$ if $f(x) = x^4 - 3x^2 + 1$

- use

$$\boxed{f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

- use template $f[\ ] = [\ ]^4 - 3[\ ]^2 + 1$

so $f(x+h) = (x+h)^4 - 3(x+h)^2 + 1$

- Now

$$f'(x) = \lim_{h \to 0} \left[\frac{[(x+h)^4 - 3(x+h)^2 + 1] - [x^4 - 3x^2 + 1]}{h} \right]$$

foil foil

$$= \lim_{h \to 0} \left[\frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - 3(x^2 + 2xh + h^2) + 1 - (x^4 - 3x^2 + 1)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - \cancel{3x^2} - 6xh - 3h^2 + \cancel{1} - \cancel{x^4} + \cancel{3x^2} - \cancel{1}}{h} \right]$$

$$= \lim_{h \to 0} \left[4x^3 + \cancel{6x^2h} + \cancel{4xh^2} + \cancel{h^3} - 6x - \cancel{3h} \right]$$

$\to 0 \quad \to 0 \quad \to 0 \quad \to 0$

$$f' = \boxed{4x^3 - 6x}$$

* We will need to learn the "tricks" to be able to evaluate some derivatives.

Ex Find $f'(x)$ if $f(x) = \frac{1}{\sqrt{2+x}}$

• use

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right],$$

• use

$$f[\ ] = \frac{1}{\sqrt{2+[\ ]}}$$

• So

$$f'(x) = \lim_{h \to 0} \left[\frac{\frac{1}{\sqrt{2+(x+h)}} - \frac{1}{\sqrt{2+x}}}{h} \right]$$

Use this: $\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}}$ shortcut cross-multiply

$$\left\{ \text{vs.} \quad \frac{a}{b}\left(\frac{d}{d}\right) + \frac{c}{d}\left(\frac{b}{b}\right) = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad+cb}{bd} \right\}$$

foil

$$f'(x) = \lim_{h\to 0}\left[\frac{1\cdot\sqrt{2+x} - \sqrt{2+(x+h)}\cdot 1}{\sqrt{2+x}\;\sqrt{2+(x+h)}\cdot h}\right] * \underbrace{\left(\frac{\sqrt{2+x} + \sqrt{2+(x+h)}}{\sqrt{2+x} + \sqrt{2+(x+h)}}\right)}_{\text{"magic one"}}$$

$$f'(x) = \lim_{h\to 0}\left[\frac{\sqrt{2+x}\,\sqrt{2+x} - \cancel{\sqrt{2+(x+h)}\,\sqrt{2+x}} + \cancel{\sqrt{2+x}\,\sqrt{2+(x+h)}} - \sqrt{2+(x+h)}\,\sqrt{2+(x+h)}}{\left\{\sqrt{2+x}\,\sqrt{2+x+h}\,\sqrt{2+x} + \sqrt{2+x}\,\sqrt{2+x+h}\,\sqrt{2+x+h}\right\}\cdot h}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{(2+x) - (2+x+h)}{\left\{(2+x)\sqrt{2+x+h} + \sqrt{2+x}\,(2+x+h)\right\}\cdot h}\right]$$

$$f'(x) = \lim_{h\to 0}\left[\frac{-\cancel{h}}{\left\{(2+x)\sqrt{2+x+\cancel{h}^{\,0}} + \sqrt{2+x}\,(2+x+\cancel{h}^{\,0})\right\}\cancel{h}}\right]$$

$$f'(x) = \frac{-1}{(2+x)\sqrt{2+x} + \sqrt{2+x}\,(2+x)}$$

$$f'(x) = \frac{-1}{2\sqrt{2+x}\,(2+x)}$$

$$\boxed{f'(x) = -\frac{1}{2(2+x)^{3/2}}}$$

II Notation

$$\boxed{f'(x) \equiv \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)}$$

- If we use $y=f(x)$ then we have

$$y' = \lim_{h\to 0}\left(\frac{y(x+h)-y(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{\overbrace{y(x+h)-y(x)}^{\Delta y}}{\underbrace{(x+h)-x}_{\Delta x}}\right)$$

$$f'(x) \equiv \lim_{\Delta x\to 0}\left(\frac{y(x+\Delta x)-y(x)}{\Delta x}\right)$$

$$\boxed{f'(x) \equiv \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}}$$

Δ's are finite differences

- This explains the use of the most common notation: $\boxed{f' \equiv \frac{dy}{dx}}$

differentials a.k.a. "infinitesmals" or ∞-small

III Differentiability

Def A function $f(x)$ is differentiable at a location $x=a$ if $f'(a)$ exisists.

Def $f(x)$ is diff'ble on an open interval if it is diff'bl at all points internal to the interval

Thm: If $f(x)$ is diff'ble at $x=a$ then $f(x)$ is continuous at $x=a$

"can't lift pencil"

Warning diff'ble $\longrightarrow$ continuous

but

continuous $\not\longrightarrow$ diff'blity.

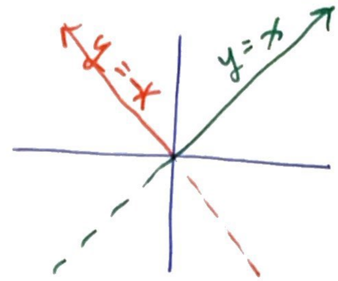
I.E. this thm is not bi conditional.

So we can't use the thm in reverse!

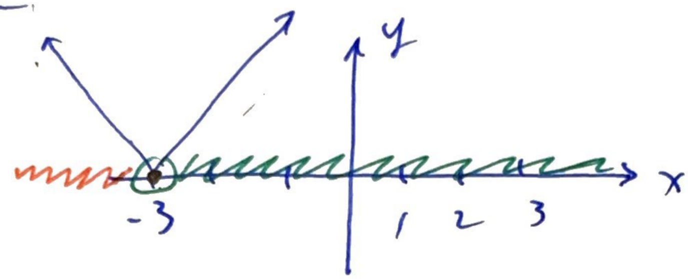
EX Where is the function $f(x) = |x+3|$ differentiable?

- Def of abs. value: $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

"vee"



- Shifting $|x+3|$



- Three cases:

(i) $\boxed{x > -3}$: $f'(x) = \lim_{h \to 0} \dfrac{|(x+h)+3| - |x+3|}{h}$

$= \lim_{h \to 0} \left(\dfrac{\cancel{x}+h+\cancel{3} - \cancel{x} - \cancel{3}}{h} \right) = \lim \left(\dfrac{\cancel{h}}{\cancel{h}} \right)^{1} = \boxed{+1}$

(ii) $\boxed{x < -3}$ $f'(x) = \lim_{h \to 0} \dfrac{|(x+h+3)| - |x+3|}{h}$

$= \lim_{h \to 0} \dfrac{\left(-(\cancel{x}+h+\cancel{3})\right) - \left(-(\cancel{x}+\cancel{3})\right)}{h} = \lim \left(\dfrac{-\cancel{h}}{\cancel{h}} \right)^{-1} = \boxed{-1}$

(iii) $\boxed{x = -3}$ Limit DNE since $\underbrace{\lim_{x \to -3^-} (\;)}_{-1} \neq \underbrace{\lim_{x \to -3^+} (\;)}_{+1}$

ANS: "f is diff'ble @" $(-\infty, -3) \cup (-3, \infty)$

a.k.a. $\boxed{\{x \mid x \neq -3\}}$

BTW f is continuous @ $x = -3$ even though f is NOT diff'bl there!

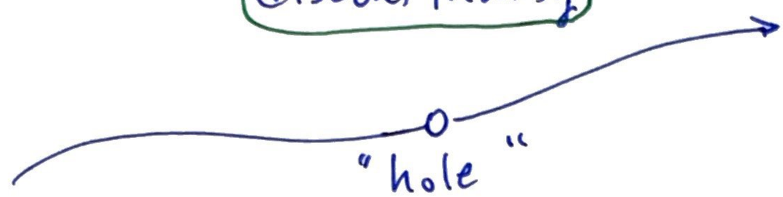
$\boxed{\text{IV}}$ Places where the derivatives fail

(a) cusp

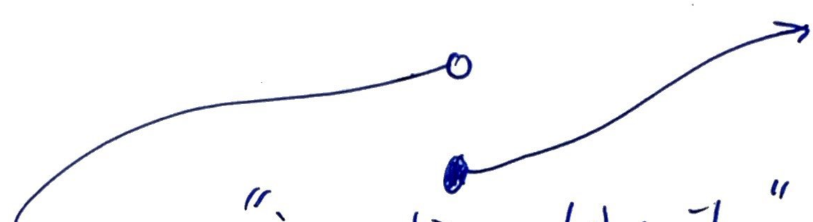
ex $f = |x|$

diff'ble everywhere but $x = a$

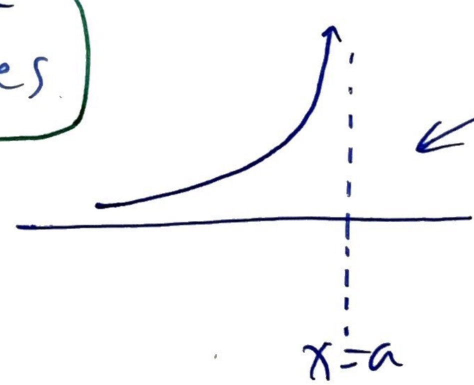
(b) discontinuity



-OR-



(c) vertical asymptotes



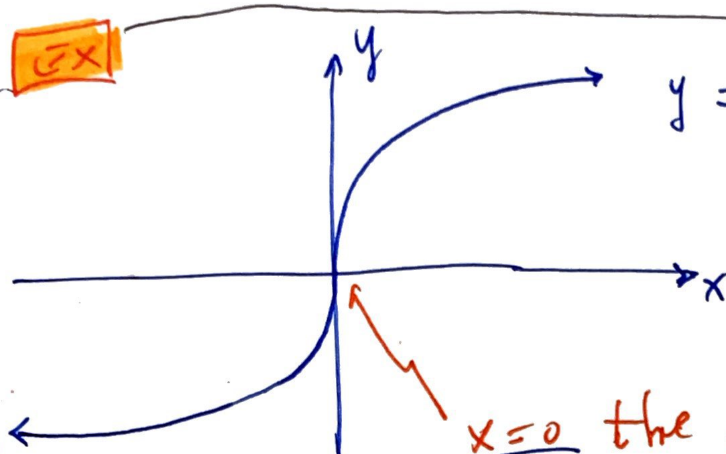
f is not diff'ble at $x = a$

f' is undefined @ $x = a$

"$\lim\limits_{x \to a} f'(x) = \infty$"

⊛ It is possible that we can have a non-cusped continuous curve and still not have diff'blity everywhere?

EX



$y = x^{1/3}$

then $y' = \frac{1}{3} x^{\frac{1}{3}-1}$

$y' = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$

x=0 the slope is $\boxed{\text{undefined}}$

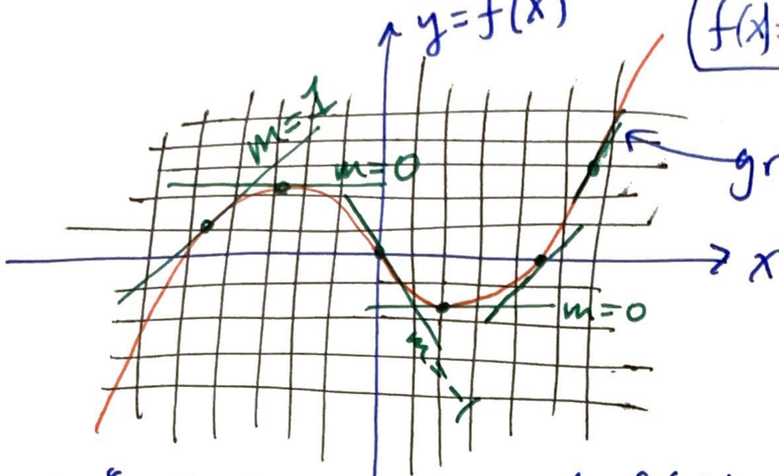
so this function, although continuous w/o a cusp, will not be diff'ble at x=0

Sketch f' given f

- Consider the graph of f(x)

function

$\boxed{f(x) = x^3 - 2x}$



graph of f(x)

cubic

$y = f(x)$

x	y

- Now ~~for~~ each "x" above, estimate f(x)'s slope as a number and plot those numbers on a separate graph:

parabola

(quadratic)