Chapter 2 The derivative

2.1 Derivatives and Rates of Change

From Chpt we discovered that the slope of the tangent line can be expressed as a limit.

$$m = \lim_{x \to a} \frac{f(a+h) - f(a)}{(a+h) - a}$$

OR Just

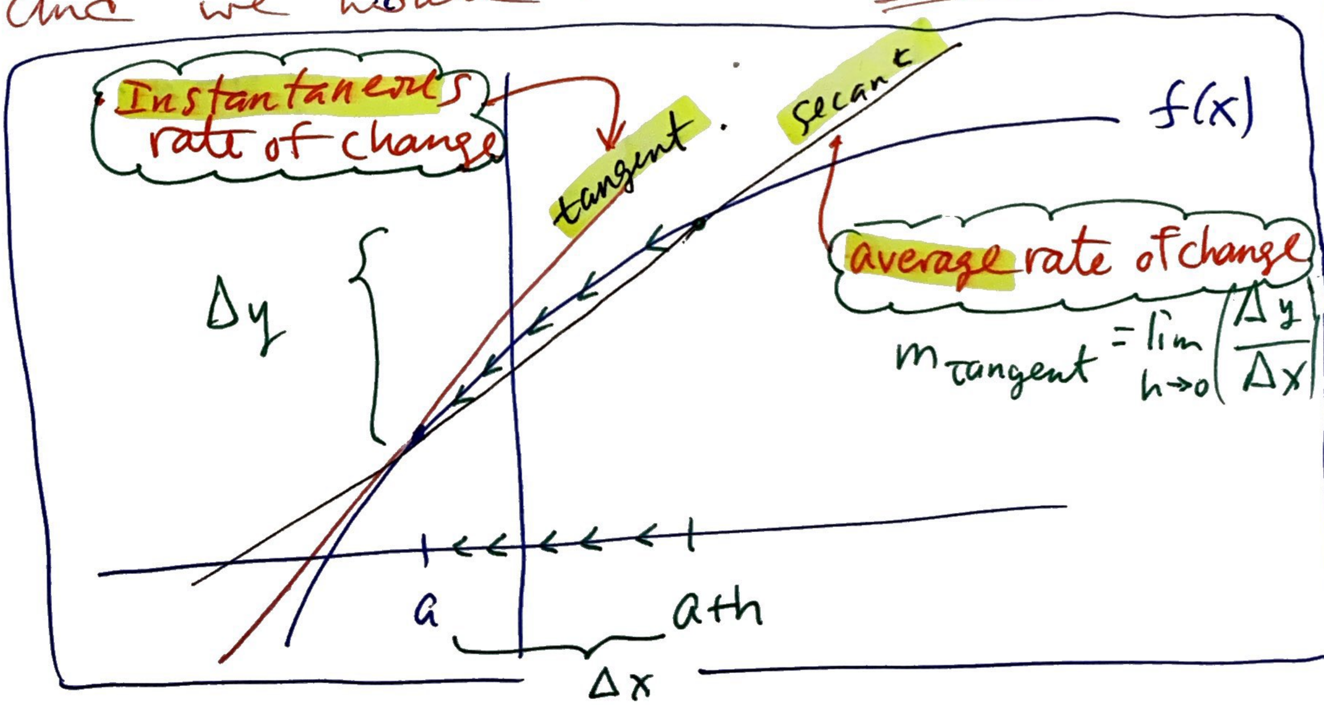
$$\boxed{m = \lim_{x \to a} \left(\frac{f(a+h) - f(a)}{h} \right)}$$

Numerically $x = 2.1$, $f(2.1)$, 2.01, $f(2.01)$, 2.001, $f(2.001)$ etc.

Calculate $\Rightarrow$ m_a, m_b, m_c

our observation

And we would notice $m_a \to m_b \to m_c \to m$



EX Find the equation of the tangent line to the graph of $y = x^2 + 2$ @ $(x,y) = (1,3)$

- let $\boxed{f(\ \ ) = (\ \ )^2 + 2}$

- feed it $1+h \Rightarrow (1+h)^2 + 2$
  feed it $1 \Rightarrow (1)^2 + 2$

$a = 1$

- $m = \lim\limits_{h\to 0} \left(\dfrac{f(a+h) - f(a)}{h} \right)$

$= \lim\limits_{h\to 0} \left(\dfrac{f(1+h) - f(1)}{h} \right)$

$= \lim\limits_{h\to 0} \left(\dfrac{[(1+h)^2 + 2] - [(1)^2 + 2]}{h} \right)$

$= \lim\limits_{h\to 0} \left(\dfrac{\cancel{1} + 2h + h^2 + \cancel{2} - \cancel{1} - \cancel{2}}{h} \right)$

$= \lim\limits_{h\to 0} \left(\dfrac{2h + h^2}{h} \right) = \lim\limits_{h\to 0} (2 + h) = \boxed{2}$

Slope of tangent line

Don't forget ... Find the eqn of the tangent line

For lines we (almost) only need $y = \underbrace{m}_{2}x + b$

Form : $y = 2x + b$

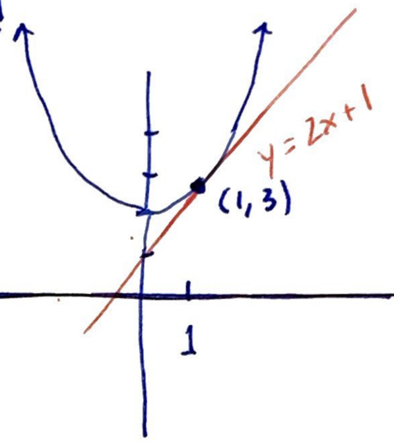
Point : $3 = 2(1) + b$

Solve : $3 - 2 = b \rightarrow b = \underline{1}$

Final : $\boxed{y = 2x + 1}$

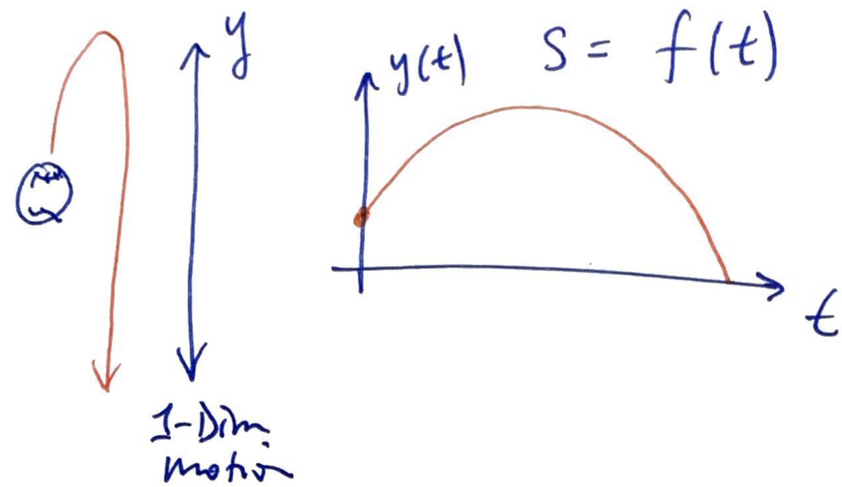
$y = x^2 + 2$

- Graph

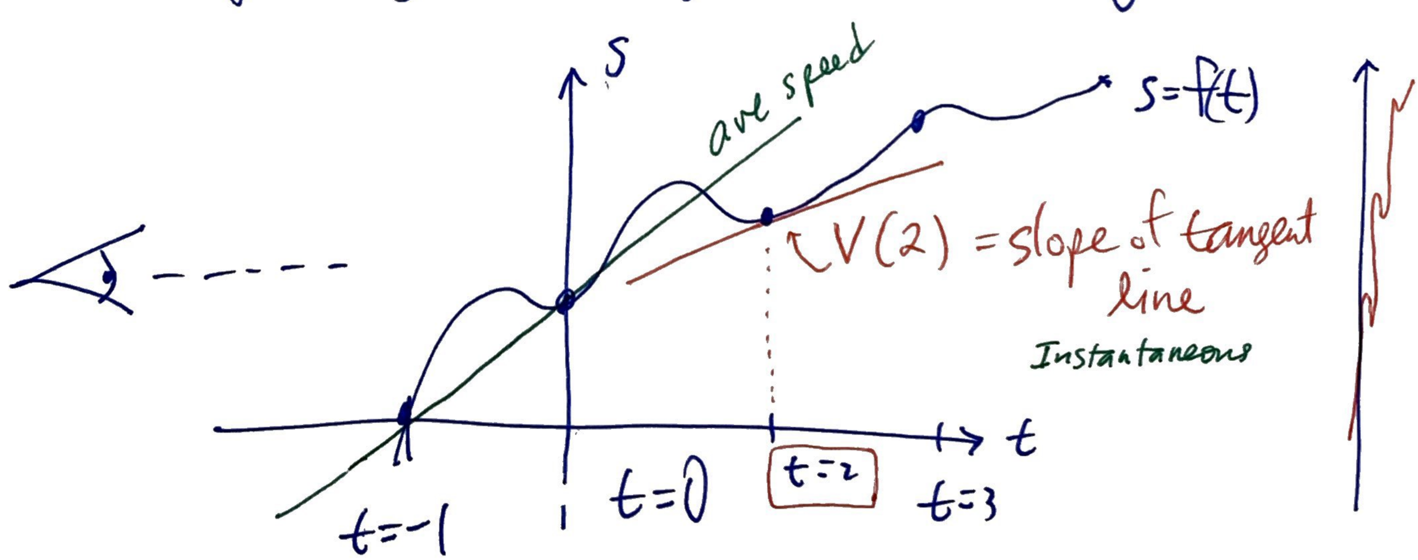


II Velocities

An object moving back and forth along a straight line may have its position described as a function



Generally we might plot experimentally, or analytically, and get something like...



velocity = instantaneous slope of this curve

Instantaneous velocity (speed)

$$V = \lim_{t \to a} \left(\frac{S_2 - S_a}{t_2 - a} \right) \qquad \boxed{\frac{\Delta s}{\Delta t} = v}$$

$$\boxed{V = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)}$$

EX let the position of a bead on a wire be described as $s(t) = t^2 - 8t + 18$

Q: Find the instantaneous speed @ $t = 3$

· let $f(\ ) = [\ \ ]^2 - 8[\ \ ] + 18$

Keeps our formula general

So

$$m = \lim_{h \to 0} \frac{\{(a+h)^2 - 8(a+h) + 18\} - \{[a]^2 - 8[a] + 18\}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{8a} - 8h + \cancel{18} - \cancel{a^2} + \cancel{8a} - \cancel{18}}{h}$$

$$= \lim_{h \to 0} \frac{2ah - 8h + h^2}{h} = \lim_{h \to 0} (2a - 8 + \cancel{h}^{\,0})$$

$$\boxed{\text{speed} = m = 2a - 8}$$

@ $t = 3$ {i.e. $a = 3$}

speed $= 2 \cdot 3 - 8$

$$\boxed{v = -2\ \text{m/s}}$$

math 211 $\boxed{\text{CW 2a}}$ Name ______

1. If speed follows $s = t^3 + 2t^2 - 9t + 5$

Find the instantaneous velocity at $t = 4$, by using a generalized formula in the time parameter "a"

$$f(\;) = (\;)^3 + 2(\;)^2 - 9(\;) + 5$$

$$v = \boxed{\lim_{h\to 0} \frac{\{(a+h)^3 + 2(a+h)^2 - 9(a+h) + 5\} - \{a^3 + 2a^2 - 9a + 5\}}{h}}$$

$$= \lim \frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 + \cancel{2a^2} + 4ah + 2h^2 - 9a - 9h + 5 - \cancel{a^3} - \cancel{2a^2} + 9a - 5}{h}$$

$$= \lim \frac{3a^2h + 3ah^2 + h^3 + 4ah + 2h^2 - 9h}{h}$$

$$= \lim_{h\to 0} \left(3a^2 + 3ah + h^2 + 4a + 2h - 9\right)$$

$$= \boxed{3a^2 + 4a - 9}$$

@ $a = 4$

$$= 3\cdot 4^2 + 4\cdot 4 - 9$$

$$= 48 + 16 - 9$$

$$= 64 - 9$$

$$= \boxed{55}$$

III Derivatives {what Math 211 is 80% about}

The derivative of a function f at $x=a$, denoted by $f'(a)$, is

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

i.e. the slope function

Ex Find the derivative of $f(x)=x^3-3x^2+1$ @ $x=a$

After chpt 2 is done $f'=3x^2-6x$ @ $x=a$
$f'(a)=3a^2-6a$.

Formal approach is

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$=\lim_{h\to 0}\frac{\{(a+h)^3-3(a+h)^2+1\}-\{a^3-3a^2+1\}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{a^3}+3a^2h+3ah^2+h^3-\cancel{3a^2}-6ah-3h^2+\cancel{1}-\cancel{a^3}+\cancel{3a^2}-\cancel{1}}{h}$$

$(a+h)^3$
$=a^3+3a^2h+3ah^2+h^3$

1
1 1
1 2 1
1 3 3 1

$$=\lim_{h\to 0}\frac{3a^2h+3ah^2+h^3-6ah-3h^2}{h}$$

$$=\lim_{h\to 0} 3a^2+3ah+h^2-6a-3h$$

$$=\lim_{h\to 0}\left(3a^2-6a+\cancel{3ah}^{\to 0}+\cancel{h^2}^{\to 0}-\cancel{3h}^{\to 0}\right)$$

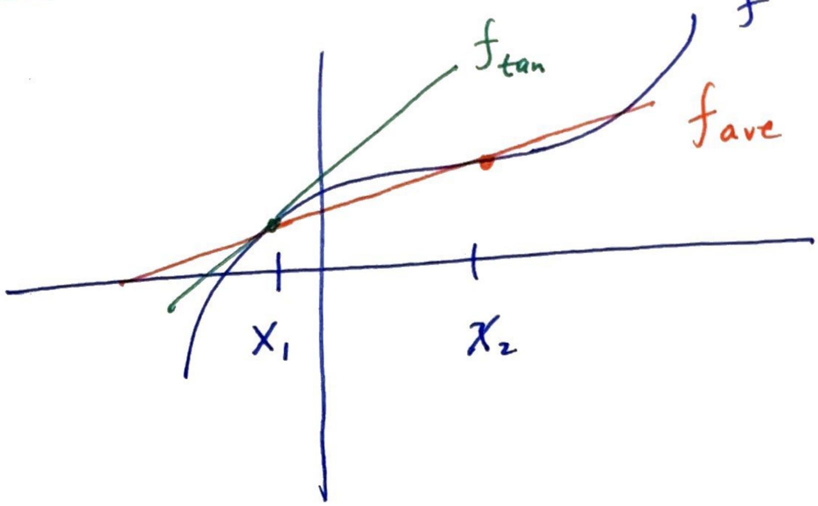
$$\boxed{f'(a)=3a^2-6a}$$

The slope of the tangent line of $f(x)$ at $x=a$.

$\boxed{\text{IV}}$ Average and

- $$\boxed{\bar{f} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$ secant line slope

- $$\boxed{f_{instant} = \lim_{h \to 0} \left(\frac{f(x_1+h) - f(x_1)}{h} \right)}$$



$\boxed{\text{Average}}$

⊛ The derivative of $f(x)$ at x is the $\boxed{\text{Instantaneous}}$ rate of change

$$\boxed{\frac{df(x)}{dx} \equiv \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)} \to \frac{0}{0}$$

- Notation

$$\text{ave } \frac{\Delta f}{\Delta x} \xrightarrow[\text{limit}]{\text{inst.}} \frac{df}{dx}$$

$$\boxed{f'(x) \equiv \frac{df}{dx}}$$ "f prime"

Ex 2 million liters of water are drained from a swimming pool a the volume

$$V(t) = 2{,}000{,}000\left(1-\frac{t}{120}\right)^2 \ell^3$$

liters cubed

Q: what is the rate at which water is leaving the pool at the 1hr mark?

(i) we seek $\dfrac{\Delta Vol}{\Delta t} \xrightarrow{h\to 0} \dfrac{dV}{dt}$ $\boxed{(a-b)^2 = a^2 - 2ab + b^2}$

(ii) $\Delta V = V(t+h) - V(t)$

$$= 2\times10^6\left(1-\frac{t+h}{120}\right)^2 - 2\times10^6\left(1-\frac{t}{120}\right)^2$$

$$= 2\times10^6\left[\left(1^2 - 2\left(\frac{t+h}{120}\right) + \frac{(t+h)^2}{120^2}\right) - \left(1-\frac{t}{120}\right)^2\right]$$

$$= 2\times10^6\left[\left(1-\frac{t}{60}-\frac{h}{60}+\frac{t^2+2th+h^2}{14400}\right) - \left(1-\frac{2t}{120}+\frac{t^2}{14400}\right)\right]$$

$$= 2\times10^6\left[\cancel{1}-\cancel{\frac{t}{60}}-\frac{h}{60}+\cancel{\frac{t^2}{14400}}+\frac{2th}{14400}+\frac{h^2}{14400}-\cancel{1}+\cancel{\frac{t}{60}}-\cancel{\frac{t^2}{14400}}\right]$$

$$\Delta V = 2\times10^6\left[-\frac{h}{60}+\frac{2th}{14400}+\frac{h^2}{14400}\right]$$

$\div h$

$$\frac{dV}{dt} = \lim_{h\to0}\left[\frac{2\times10^6\left[-\frac{\cancel{h}}{60}+\frac{2t\cancel{h}}{14400}+\frac{h^2}{14400}\right]}{\cancel{h}}\right]$$

$$\frac{dV}{dt} = 2\times10^6\left[\frac{2t}{14400}-\frac{1}{60}\right]$$

$$~~\frac{dV}{dt}~~\quad V' = \boxed{\frac{2500}{9}(t-120)}$$

@ $t = 60_{min}$ $\quad \dfrac{2500}{9}(60-120) = -16\,666\,\tfrac{2}{3}\ \ell/min$