$\boxed{1.8}$ Continuity

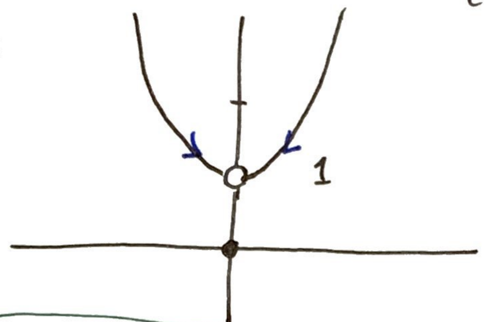
$\boxed{I}$ Continuity

Def: $f(x)$ is continuous at "a" if

$$\lim_{x \to a} f(x) = f(a)$$

Conterexample : $f(x) = \begin{cases} x^2+1 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

graph :



$\lim_{x \to 0} f(x) = 1$

$\Rightarrow$ $x \to 0^-$ & $x \to 0^+$ same

But $\lim_{x \to 0} f(x) = 1$ is not $\neq$ $f(0)$ which is 0

This function is not continous

Proving continuity involves three steps:

(a) $f(a)$ needs to be defined at $x = a$
{ "a" is in f's domain }

(b) $\lim_{x \to a} f(x)$ exist.

(c) $\lim_{x \to a} f(x) = f(a)$

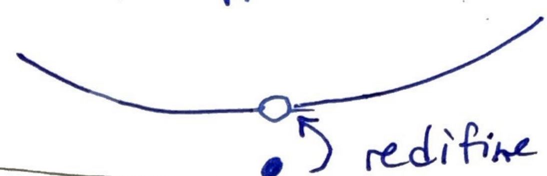
part c) failed in the opening example.

Def: $f(x)$ is discontinuous at $x=a$ if $f(x)$ is not continuous @ $x=a$

⊛ Types of Discontinuities

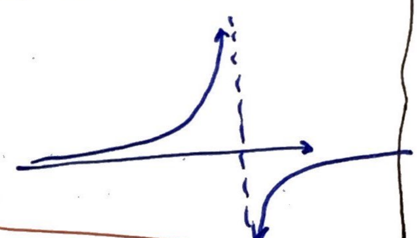
(i) "Removable": we can redifine a value of $f(x)$ at $x=a$ so that the LH approach is equal to the right hand approach

a.k.a. "Hole discontinuity"



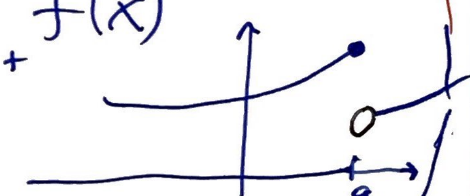
(ii) "Infinite" discontinuity:

These are locations where a vertical asymptote exisists.

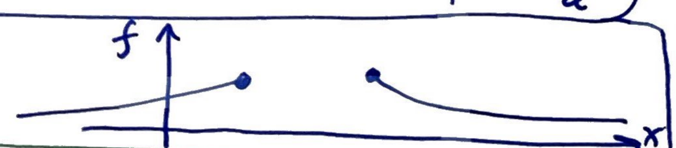


(iii) "Jump" discontinuity:

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$



(iv) "Gap" discontinuity:



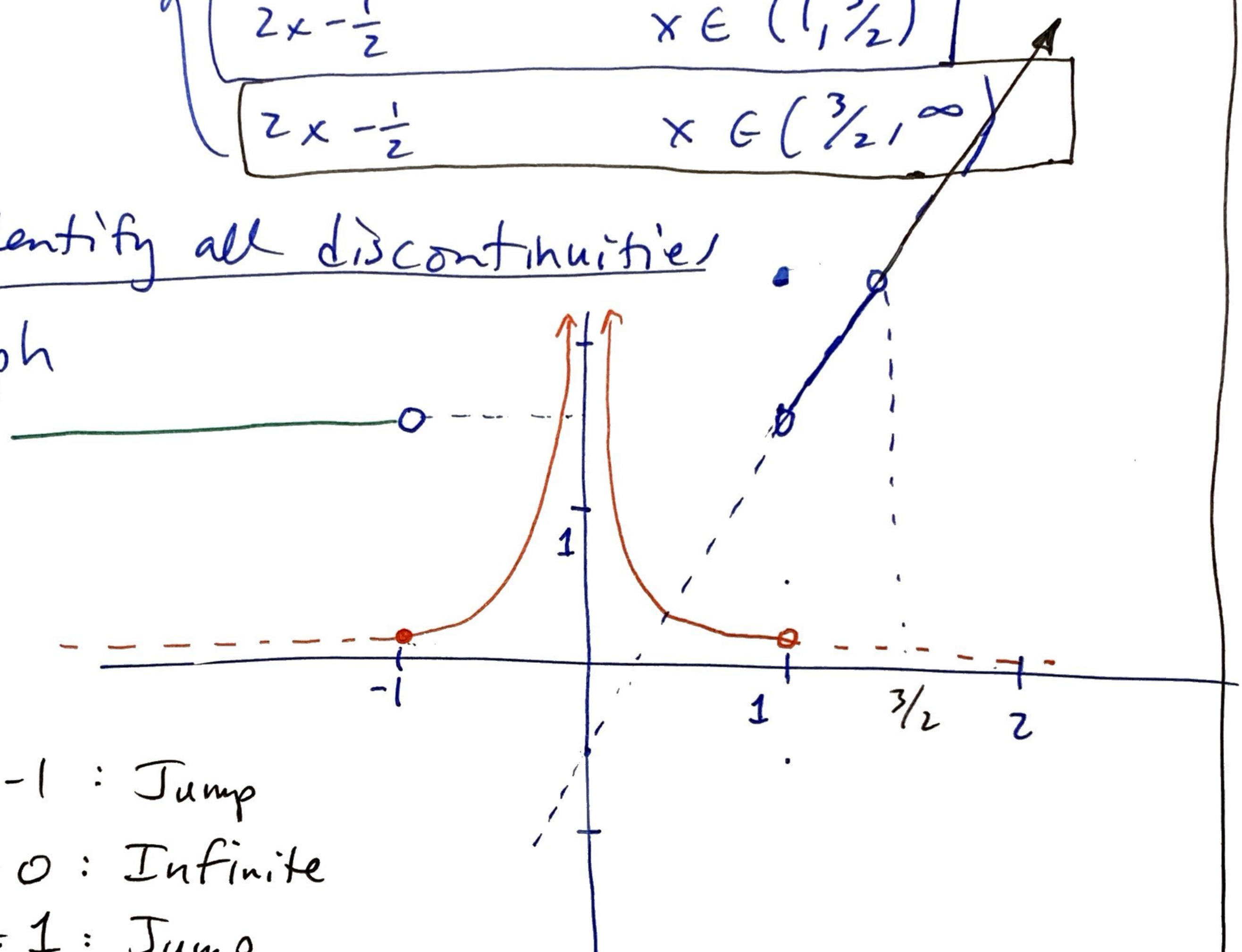
"A function is discontinuous if you need to lift your pencil while tracing the graph of the function"

EX Consider the piecewise function

$$f(x) = \begin{cases} 3/2 & x < -1 \\ 1/x^2 & [-1,0) \cup (0,1) \\ 5/2 & x = 1 \\ 2x - \frac{1}{2} & x \in (1, 3/2) \\ 2x - \frac{1}{2} & x \in (3/2, \infty) \end{cases}$$

Identify all discontinuities

Graph



@ $x = -1$: Jump

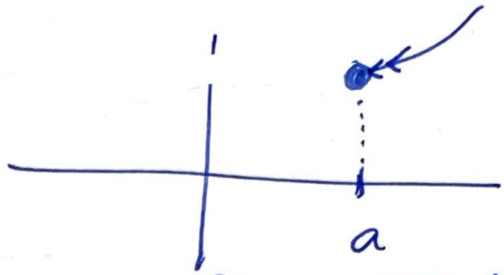
@ $x = 0$: Infinite

@ $x = 1$: Jump

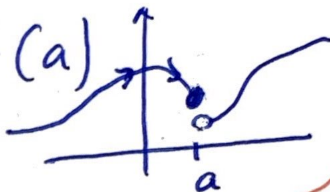
@ $x = \frac{3}{2}$: Hole {removable}

✱ One-sided continuity

Def: $f(x)$ is continuous from the right at $x=a$ if $\lim_{x\to a^+} f(x) = f(a)$



Def: $f(x)$ is continuous from the left @ $x=a$ if $\lim_{x\to a^-} f(x) = f(a)$



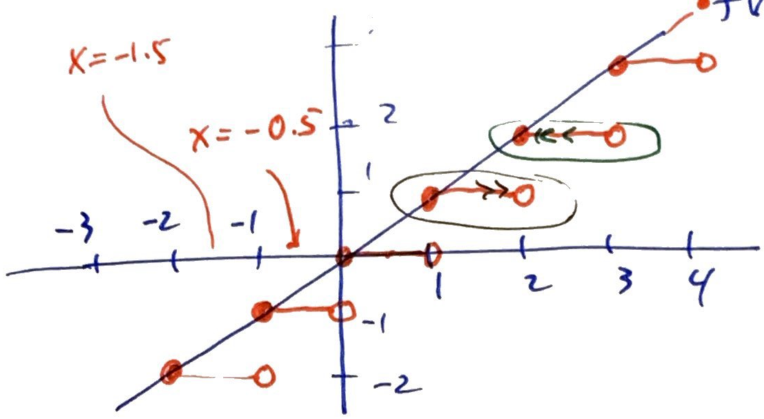
Def: $f(x)$ is continuous on an interval if it is continuous at all interior points in the interval

If at a closed boundary at an edge of the interval AND if $f(x)$ is one-sided continuous at that closed boundary then we include the boundary in the interval

EX Step function

$g(x) = [\![x]\!]$ truncate a decimal number's value to ~~the~~ next lowest integer

$f(x) = x$, $g(x) = [\![x]\!]$



$g(0.5) = [\![0.5]\!]$
$= 0$

$g(0) = [\![0]\!]$
$= 0$

$g(1) = [\![1]\!] = 1$

$g(1.5) = [\![1.5]\!] = 1$

$g(-1.5) = 2$

- consider $x \in [2, 3)$ (element of)

$[\![x]\!]$ is continuous from the right

b/c $\lim_{x \to 2^+} [\![x]\!] = 2$ itself.

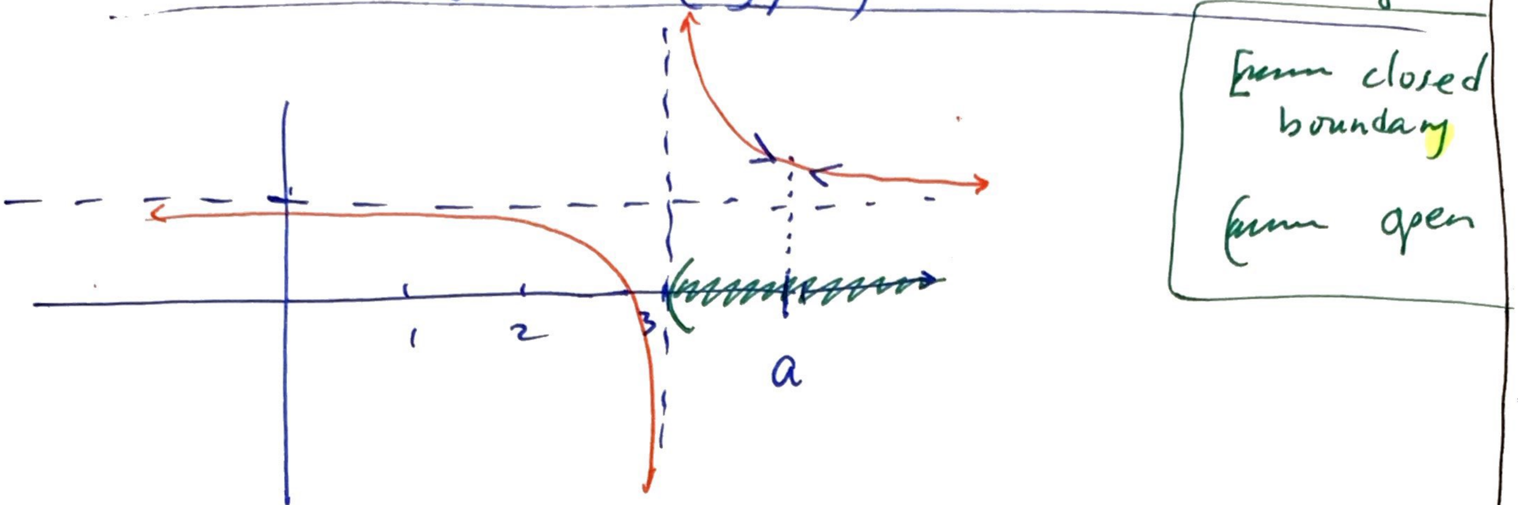
but $[\![x]\!]$ is not continuous from the left

b/c $\lim_{x \to 2^-} [\![x]\!] \neq 2$ itself but rather is equal to 1

In general $\lim_{x \to 2} g(x)$ does not exist (DNE)

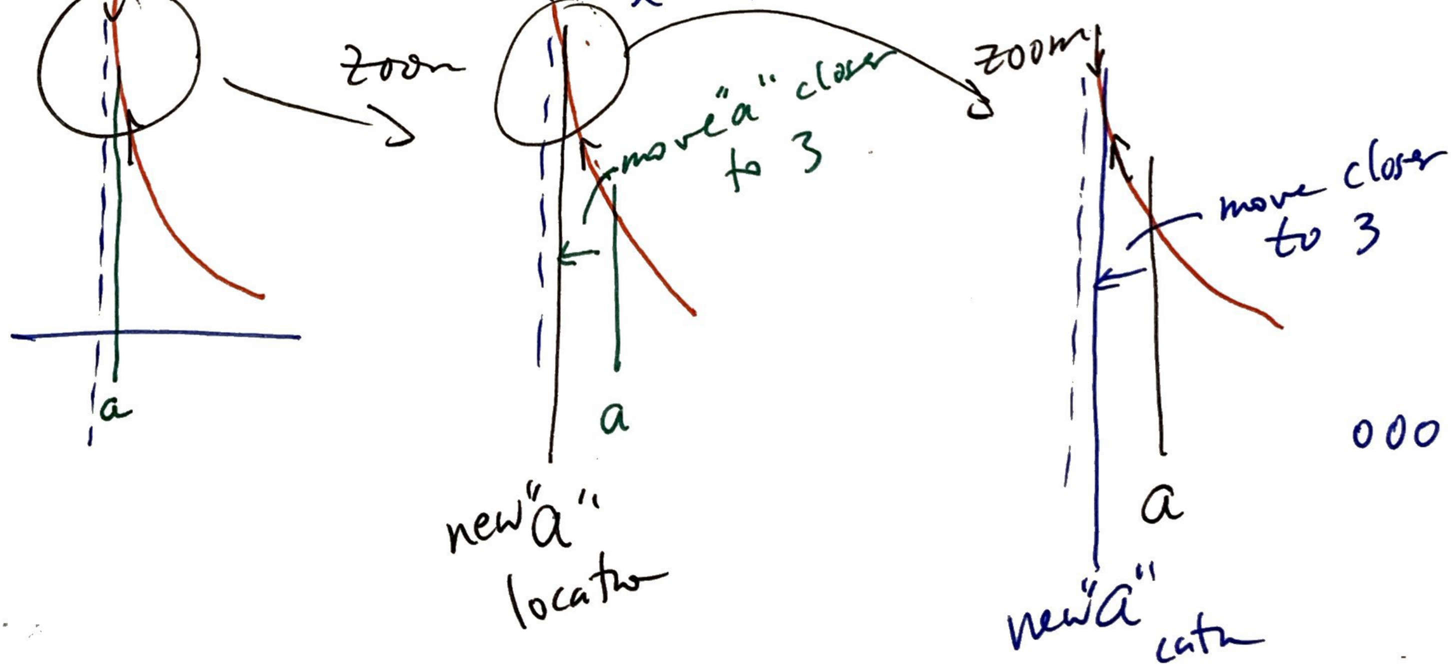
since $\lim_{x \to 2^-} g(x) \neq \lim_{x \to 2^+} g(x)$

EX Prove $f(x) = \left(\frac{x+1}{x-3}\right)$ is continuous on the interval $(3, \infty)$



For all $a > 3$ $\quad \lim\limits_{x \to a^-} \left(\frac{x+1}{x-3}\right) = \lim\limits_{x \to a^+} \left(\frac{x+1}{x-3}\right)$

at the boundary, open, we can push our "a" as close to 3 as we desire and always have $\lim\limits_{x \to a^-} = \lim\limits_{x \to a^-}$



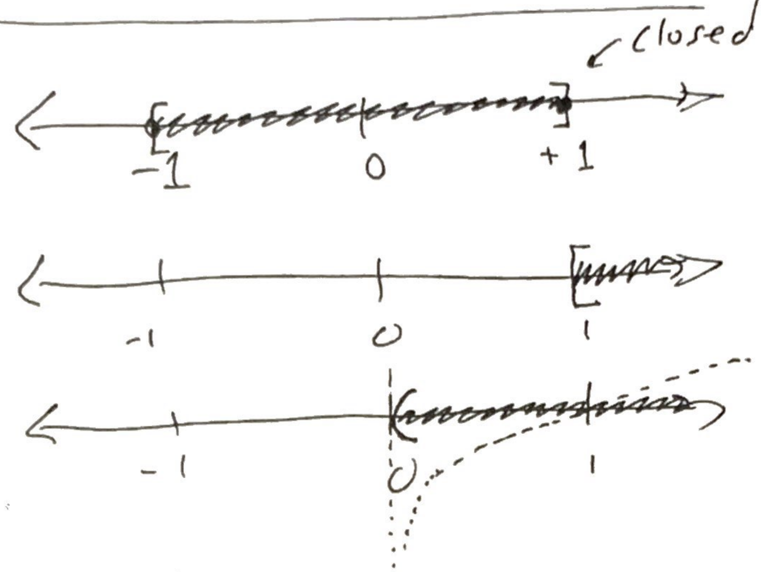
EX where is the function
$$f(x) = \frac{\sin^{-1}(x) + \sqrt{x-1}}{\ln(x)}$$
continuous?

(1st)

domain of $\sin^{-1}(x)$

domain of $\sqrt{x-1}$

domain of $\ln(x)$



however we cannot include $x=1$ in the domain of $f(x)$, above, since $\ln(1)=0$ & we ÷ 0

(2nd)

- We examine the intersection of all 3 pieces and we see that only $f(1)$ would be defined in the numerator and also in the denominator but since the denominator is zero at $x=1$ we need to exclude that point also so...

Bottom Line: $\boxed{\text{This function is continuous Nowhere!}}$

BTW: had we multiplied then

{ $\left(\sin^{-1}(x) + \sqrt{x-1}\right) * \ln(x)$ exists at $x=1$, but nowhere else. (it is not continuous there) }

II Properties

IF f and g are continuous at $x=a$ and if "c" is a constant then

$\boxed{f+g\ ,\ f-g}$, $\boxed{f\cdot g\ ,\ \frac{f}{g}}$, $\boxed{cf}$ are all continuous at $x=a$ also $\{g(a)\neq 0\}$

EX: all polynomials are continuous on $(-\infty,\infty)$

EX: all rational functions are continuous on $(-\infty,\infty)$ except where the denominator vanishes

In their respective domains {where they are defined} the following are continuous

- polynomials
- rational functions
- root functions $\sqrt[n]{(\ )}$
- trig functions:
- inverse trig functions
- exponential functions
- logarithmic functions

- If $f(x)$ is continuous at $x=b$
  and if $\lim_{x\to a} g(x) = b$
  then $\lim_{x\to a} f(g(x)) = f(b)$

{ composites of continuous outer functions are continuous where defined }

$$\boxed{f\circ g \equiv f(g(x))}$$

Then $\lim_{x\to a} f\circ g = f(b)$ where $g(a)=b$

- If g is continuous at $x=a$ and if f is continuous at $g(a)$ then the composite function $f\circ g$ is also continuous @ $x=a$

EX evaluate $\lim_{x \to 4} \ln\left(\frac{x^2-16}{x-4}\right)$ h

In pre-calc. class we exclude $x=4$ from the domain

in calc. $\lim_{x \to 4} \ln\left(\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}}\right)$

$= \lim_{x \to 4} \ln((x+4))$ g f

$= \ln(\lim_{x \to 4} x+4)$

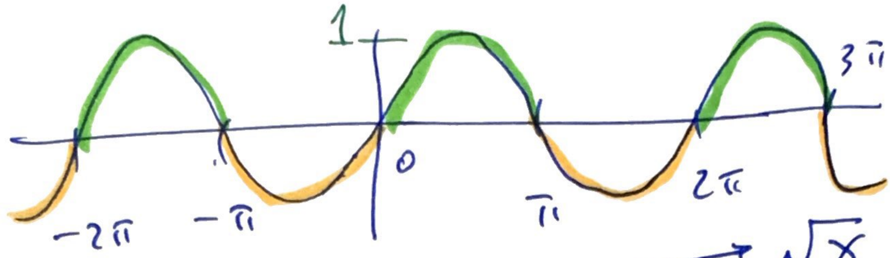
$= \ln(4+4) = \ln(8) = \boxed{3\ln(2)}$

11

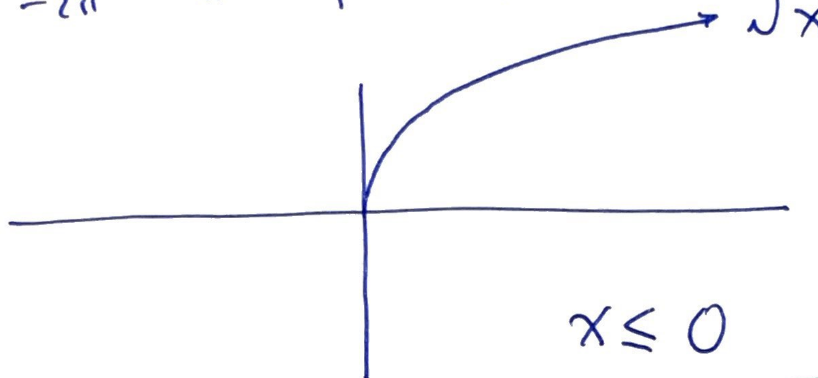
EX Where is $f(x) = \sqrt{\sin(x)}$ continuous?

decompose $h \circ g$
$$\begin{cases} h(\ ) = \sqrt{(\ )} \\ g(\ ) = \sin(\ ) \end{cases}$$

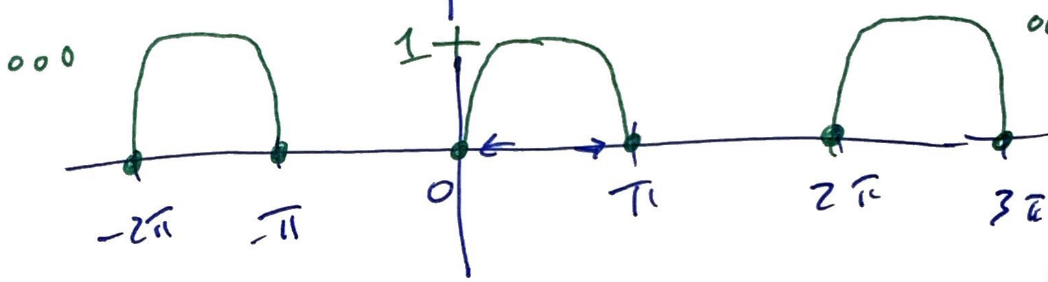
- $\sin(x)$
  $(-\infty, \infty)$



- $\sqrt{x}$
  $[0, \infty)$



Together



- To be continuous we need $f(a)$ defined
  and we need $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$
  which is fine for all x $\ni$ (such that) $\boxed{2\pi k < x < (2k+1)\pi}$
  for $k \in \mathbb{Z}$ {integers : ...−2, −1, 0, 1, 2, ...}
- However we can also include the boundaries since we have one-sided continuity

$$\boxed{2\pi k \leq x \leq (2k+1)\pi}$$

Note
$2k$ = even always
$2k+1$ = odd always

Intermediate Value Theorem

Thm: If $f(x)$ is continuous on the closed interval $[a,b]$

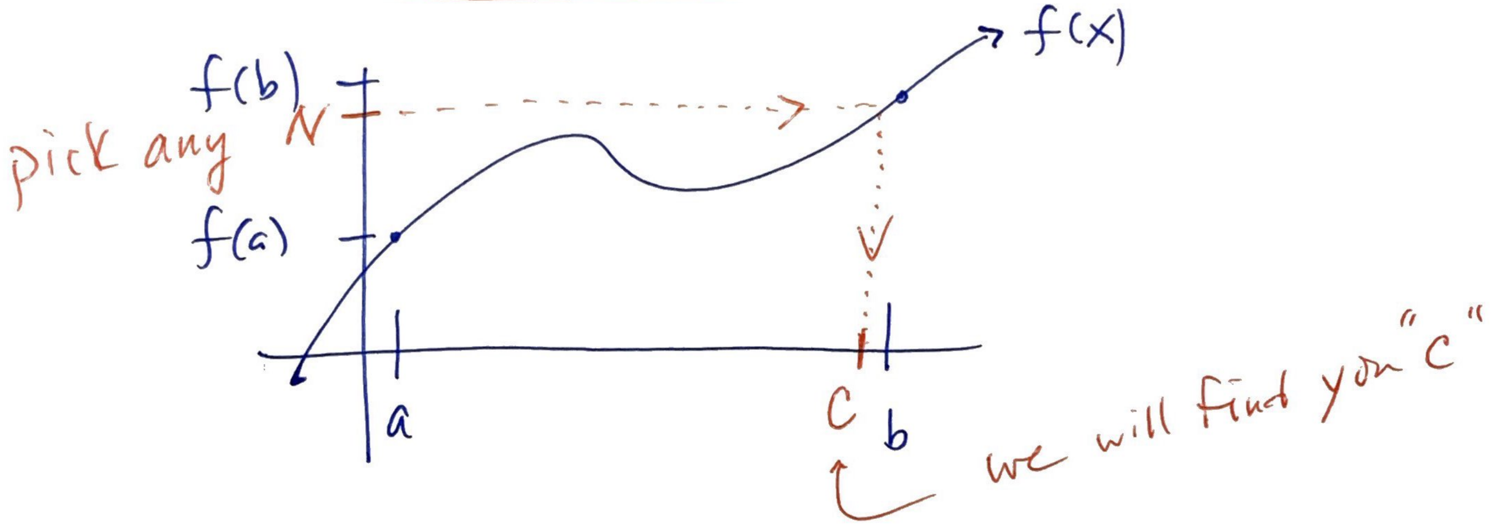
then for any number $N \in [f(a), f(b)]$

there exists a number "c" $\in [a,b]$

such that

$$f(c) = N$$

$f(a) \neq f(b)$



Math Notation

$\in$ "element of"

∂ "boundary"

$\exists$ "there exists"

$\forall$ "for all"

"$|$" or $\ni$ "such that"

Conditionals

"a" implies "b"
aka
if "a" then "b"

$a \rightarrow b$

iff

Bi-conditional

$a \rightarrow b$ & $b \rightarrow a$

notation: $a \leftrightarrow b$

"a" if and only if "b"

* Theorem Structure

- premise 1 { conditions or constraints }
- premise 2 { }

⋮

"imply" → ⟹ conclusion

Logically

$$\underbrace{(P_1 \text{ and } P_2 \text{ and } \cdots P_n)}_{\substack{\text{conditions} \\ \text{or} \\ \text{assumptions} \\ \text{(postulates)}}} \rightarrow \text{conclusion}$$

imply

again but in "Math speak"

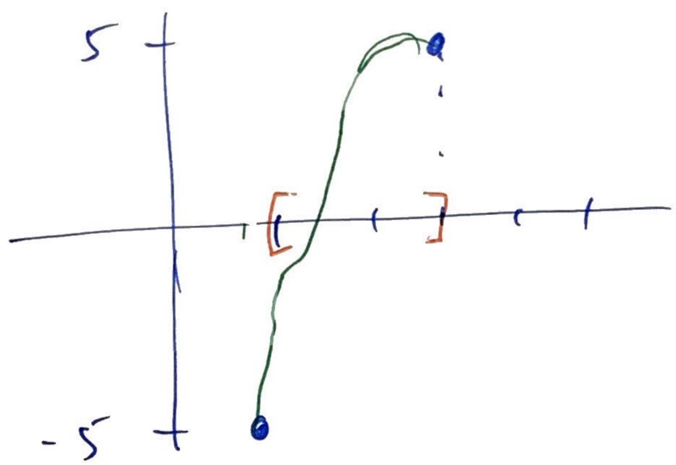
I.V.Thm

for f, C_0 on $[a,b]$,

$\rightarrow \forall N \in [f(a), f(b)]\ \exists\ c \in [a,b]$

$\ni f(c) = N$

EX If $f(1) = -5$ and $f(3) = +5$ does there have to be an x in $[1,3]$ such that $f(x) = 0$



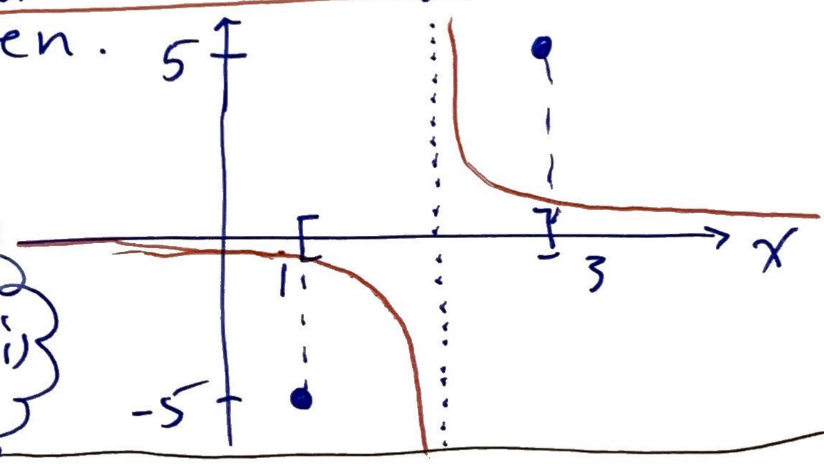
pick $N=0$

you must find me $c \ni f(c) = 0$

Q: before we invoke the IVT is the condition (premise) met?

A: No mention of continuity! So I can lift my pen.

Counter example



No... we need not have $f(x) = 0$ in this int'vl $[1,3]$

* office hours

48.

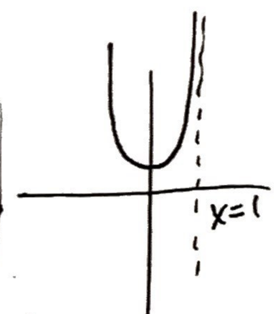
$$m = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\lim_{v \to c^-} \left(\frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \right)$$

As $v \to c^-$ then $\left(\frac{v}{c}\right) \to 1$ and $\sqrt{1-\frac{v}{c}} \to 0$

So the limit is ∞. "undefined"

Desmos $\dfrac{1}{\sqrt{1-x^2}}$ $x \to 1^-$



$f(0.9)$, $f(0.99)$, $f(0.999)$ ...

49. (a) Use numerical. (skip)