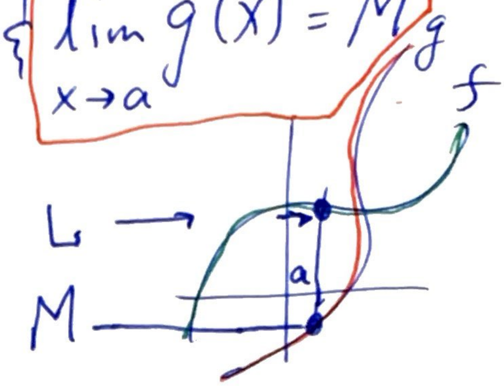
# 1.6 Limit Properties

## I Laws

let $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} g(x) = M$




sum rule: $$\lim_{x \to a} \left[ f(x) + g(x) \right] = L + M$$

diff'nce rule: $$\lim_{x \to a} \left[ f(x) - g(x) \right] = L - M$$

a.k.a. $\lim_{x \to a} \left[ f \pm g \right] = L \pm M$

scalar mult: $$\lim_{x \to a} \left[ c f(x) \right] = c \lim_{x \to a} f = c \cdot L$$

product rule: $$\lim_{x \to a} f \cdot g = \left( \lim_{x \to a} f \right) \cdot \left( \lim_{x \to a} g \right) = L \cdot M$$

goutient rule: $$\lim_{x \to a} \left( \frac{f}{g} \right) = \frac{\lim f}{\lim g} = \frac{L}{M}$$ so long as $M \neq 0$

power rule: $$\lim_{x \to a} \left[ f(x) \right]^n = \left[ \lim_{x \to a} f(x) \right]^n = L^n$$

root law: $$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f} = \sqrt[n]{L}$$

**EX** $\lim_{x \to -2} \sqrt{x^4 + 3x + 6}$ justify each step

$= \sqrt{\lim_{x \to -2} (x^4 + 3x + 6)}$ root rule

add'n rule

$= \sqrt{\lim_{x \to -2} (x^4) + \lim_{x \to -2} (3x) + \lim_{x \to -2} (6)}$

power  scalar mult.

$= \sqrt{\left[\lim_{x \to -2} (x)\right]^4 + 3 \lim_{x \to -2} (x) + 6 \lim_{x \to -2} (1)}$ eval limits

$= \sqrt{[-2]^4 + 3[-2] + 6 \cdot [1]}$ direct subst.

$= \sqrt{16 + (-6) + 6} = \boxed{4}$

since all parts are well behaved.

**(*) Direct Substitution Rule**

If $f(x)$ is a polynomial or a rational function {ratio of polynomials} then if "a" is f(x) domain, i.e. $D_f$: {contains "a"}

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

**Ex** Evaluate $\lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3)$ ③

$$= \lim_{x \to -1} (x^4 - 3x) \cdot \lim_{x \to -1} (x^2 + 5x + 3)$$

$$= \left[(-1)^4 - 3(-1)\right] \cdot \left[(-1)^2 + 5(-1) + 3\right]$$

$$= \big(1 + (3)\big) \cdot (-1) = \boxed{-4}$$

**Ex** Eval $\lim_{x \to 4} \left(\frac{x^2 - 4x}{x^2 - 3x - 4}\right)$ ← rational function so just plug in $x = 4$: $\frac{0}{0}$

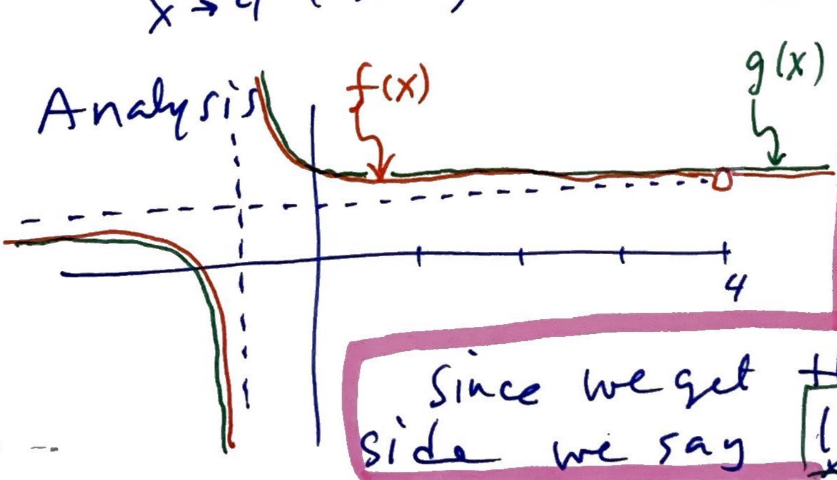
4 is NOT in the domain of $\left(\frac{x^2 - 4x}{x^2 - 3x - 4}\right)$

A different approach is needed

- factor $\underbrace{\frac{x(\cancel{x-4})}{(x+1)(\cancel{x-4})}}_{\text{let be } f(x)} = \underbrace{\frac{x}{x+1}}_{\text{call this } g(x)}$

- $\lim_{x \to 4} \left(\frac{x}{x+1}\right) = \frac{4}{4+1} = \boxed{4/5}$

Analysis




$\lim_{x \to 4^-} f(x) = \frac{4}{5}$

but $\lim_{x \to 4^+} f(x) = 4/5$

Since we get ~~the same limit~~ from either side we say $\boxed{\lim_{x \to 4} f(x) = \frac{4}{5}}$

EX Evaluate $\lim_{x \to -3} \left( \dfrac{3-|x|}{3+x} \right) \leftarrow f$

face value: $\dfrac{3-|-3|}{3+(-3)} = \dfrac{3-3}{3-3} = \dfrac{0}{0}$

So $-3$ is NOT in the domain of $\dfrac{3-|x|}{3+x}$.

Lets look at the one-sided limits

$x \to -3^-$ and $x \to -3^+$

Review

$$|x| = \begin{cases} +x & \geq 0 \\ -x & < 0 \end{cases}$$

when $x$ is negative then negate the value

$|-3| = -(-3) = 3$

$|+3| = +(+3) = 3$

- $\lim_{x \to -3^-} \left( \dfrac{3-|x|}{3+x} \right)$

$= \lim_{x \to -3^-} \left( \dfrac{3-(-x)}{3+x} \right)$

$= \lim_{x \to -3^-} \left( \dfrac{\cancel{3+x}}{\cancel{3+x}} \right) = \boxed{1}$

- $\lim_{x \to -3^+} \left( \dfrac{3-|x|}{3+x} \right)$

$= \lim_{x \to -3^+} \left( \dfrac{3-(-x)}{3+x} \right)$

$= \lim_{x \to -3^+} \left( \dfrac{\cancel{3+x}}{\cancel{3+x}} \right) = \boxed{1}$

since $\lim_{x \to -3^-} f = 1$

and $\lim_{x \to -3^+} f = 1$

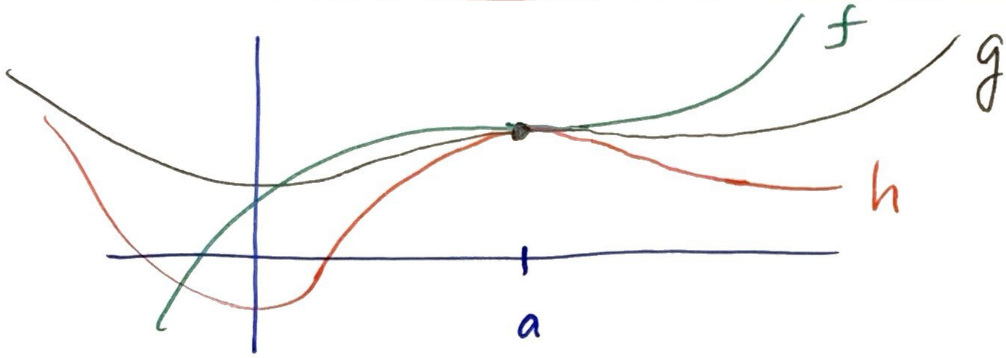
Then $\boxed{\lim_{x \to -3} \left( \dfrac{3-|x|}{3+x} \right) = 1}$

## II Squeeze Thm

If $f(x) \leq g(x) \leq h(x)$ near $x = a$,
{except possibly at "a" itself}

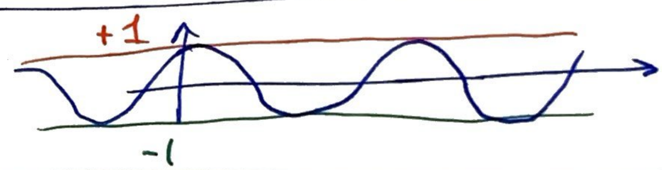
AND if $\boxed{\lim_{x\to a} f = L}$ and $\boxed{\lim_{x\to a} h = L}$ also

then $\boxed{\lim_{x\to a} g(x) = L}$ too



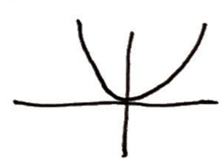

**EX** Eval $\lim_{x\to 0} \left(x^2 \cos(x)\right)$ using the range of cos

$R_{y=\cos(x)} : \{ y \mid |y| \leq 1 \}$




BTW $|y| \leq 1$ is a shortcut to $-1 \leq y \leq 1$

- Near $x=0$ $x^2$ is positive



$x^2 * \left[ \boxed{-1 \leq \cos(x) \leq +1} \right]$

$\Rightarrow \quad -x^2 \leq x^2 \cos(x) \leq x^2$

$\downarrow$ 0 by m $^2\cos$ $\downarrow$ 0

$\boxed{\lim_{x\to 0} x^2 \cos(x) = 0}$