1.5 Limits of a function

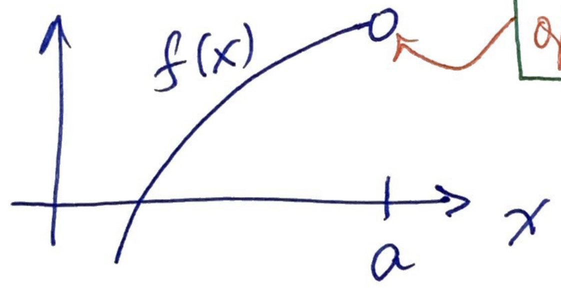
The function $f(x)$ is defined @ $x=a$ If we can calculate $f(a)$.

EX $\tan\left(\frac{\pi}{4}\right)=1$ $\tan\left(\frac{\pi}{2}\right)=$ undefined
∞

tan is defined @ $\frac{\pi}{4}$ but not $\frac{\pi}{2}$.

- It is possible that we cannot evaluate a function at $x=a$ but we will see that as we get closer to $x=a$ we can see the function's value converge.

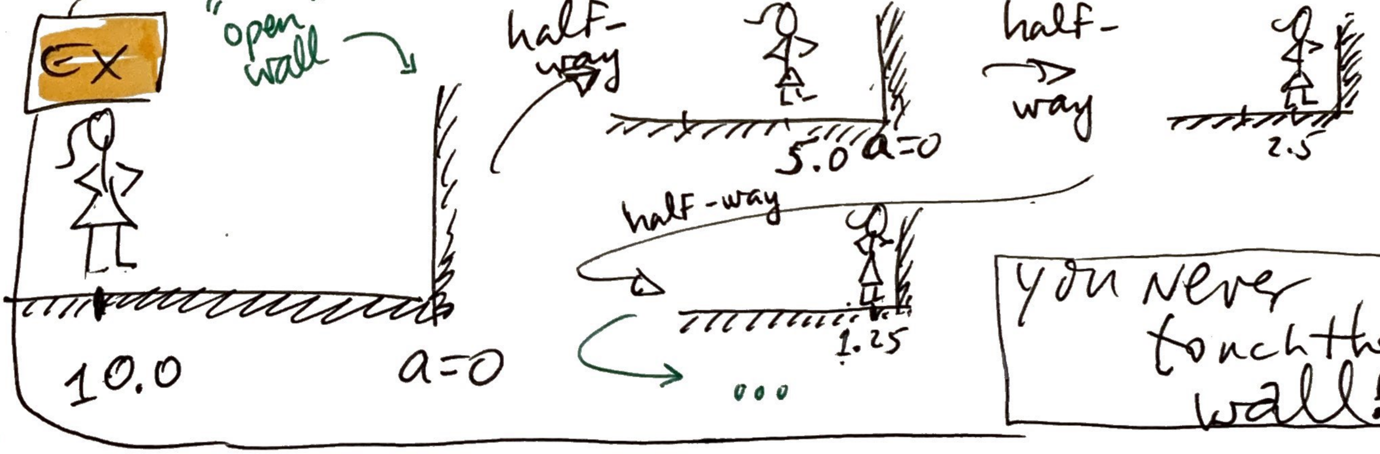
EX



open interval at $x=a$

- we can get closer and closer but never touch $x=a$

EX

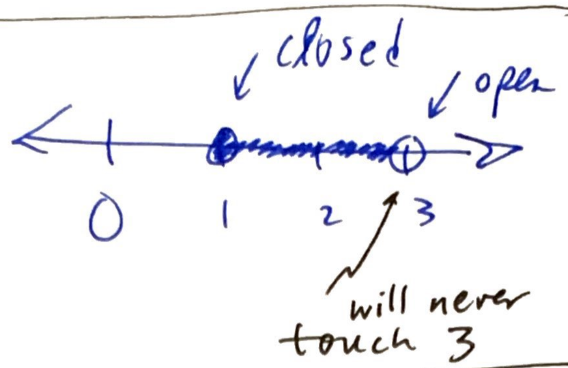


you never touch the wall!

EX Number lines

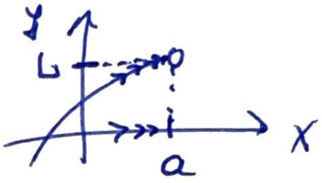
$2 \le x < 3$

↑ closed ↖ open



⊛ Since we can evaluate f(x) for all values approaching x = a we can surmise the f(x) has a limit

$$\boxed{\lim_{x \to a} f(x) = L}$$



EX Estimate the limit as x → 0 for the function $f(x) = \frac{e^{4x} - 1}{x}$

x	$f(x) = \frac{e^{4x} - 1}{x}$	
0.1	$f(0.1) = \frac{\exp(4(0.1)) - 1}{0.1}$	$= \frac{1.491825 - 1}{0.1}$ $= 4.91825$
0.01	$f(0.01) = \frac{\exp[4(0.01)] - 1}{0.01}$	$= 4.0811$
0.001	$f(0.001) = \frac{\exp[4(0.001)] - 1}{0.001}$	$= \boxed{4.00801}$
-0.1	$f(-0.1) = 3.29680$	
-0.01	$f(-0.01) = 3.921056$	
-0.001	$f(-0.001) = \boxed{3.99201}$	

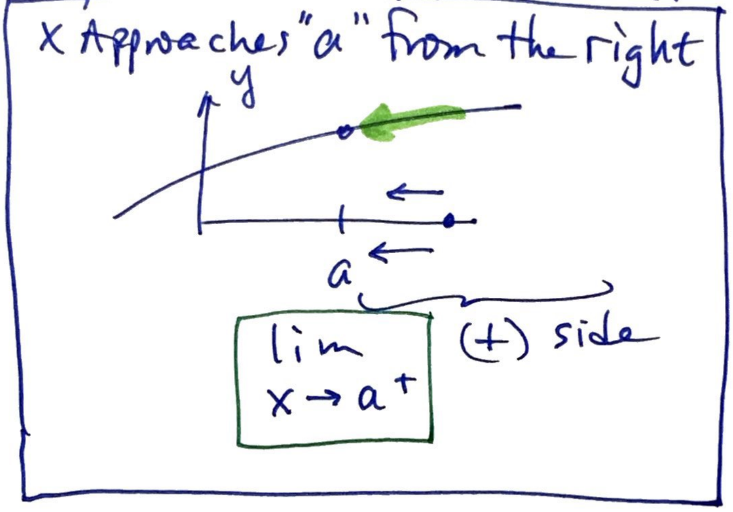
Summary

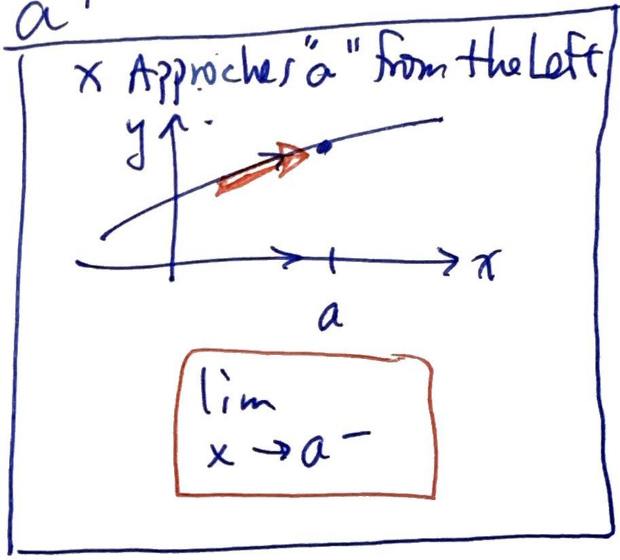
$$\boxed{\lim_{x \to 0} \frac{e^{4x} - 1}{x} = 4}$$

Soon we will introduce the derivative of a function $f(x)$, evaluated at $x = a$, as

$$\frac{\Delta f}{\Delta x} \to \boxed{\frac{df}{dx} \equiv \lim_{x\to a} \frac{f(x)-f(a)}{x-a}} \begin{matrix} \to 0 \\ \hline \to 0 \end{matrix}$$

For $f(x)$ to have a limit as $x \to a$ then we say $f(x)$, as $x \to a^-$, must equal $f(x)$ as $x \to a^+$





Then a two sided limit exists if, by definition

we have $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

or we just say "the limit exist"

- Notationally

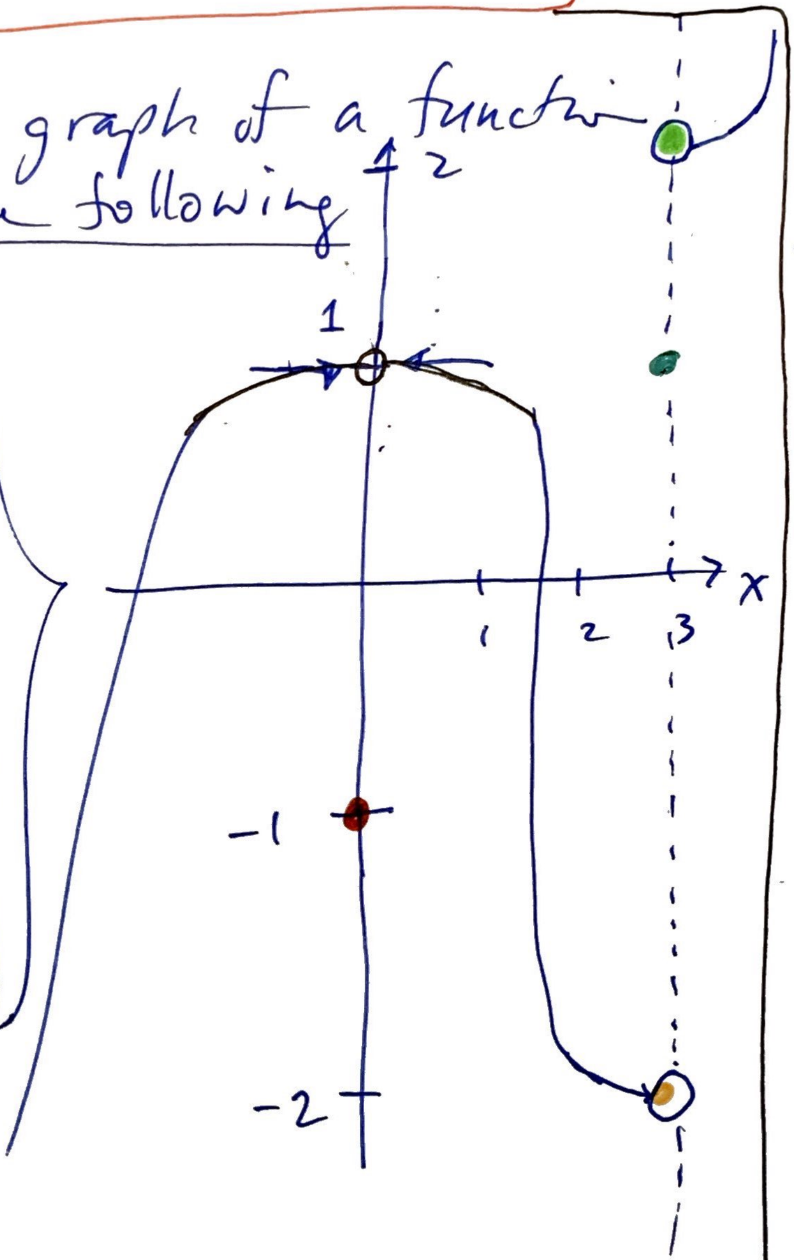
$$\boxed{\lim_{x\to a} f(x) = L \text{ means both } \lim_{x\to a^-} f(x) = L \text{ \& } \lim_{x\to a^+} f(x) = L}$$

$\boxed{II}$ One-sided Limits

- Left-handed limit: $\boxed{\lim_{x \to a^-} f(x) = L}$
- RH limit: $\boxed{\lim_{x \to a^+} f(x) = M}$

$\boxed{EX}$ Sketch a possible graph of a function that satisfies the following

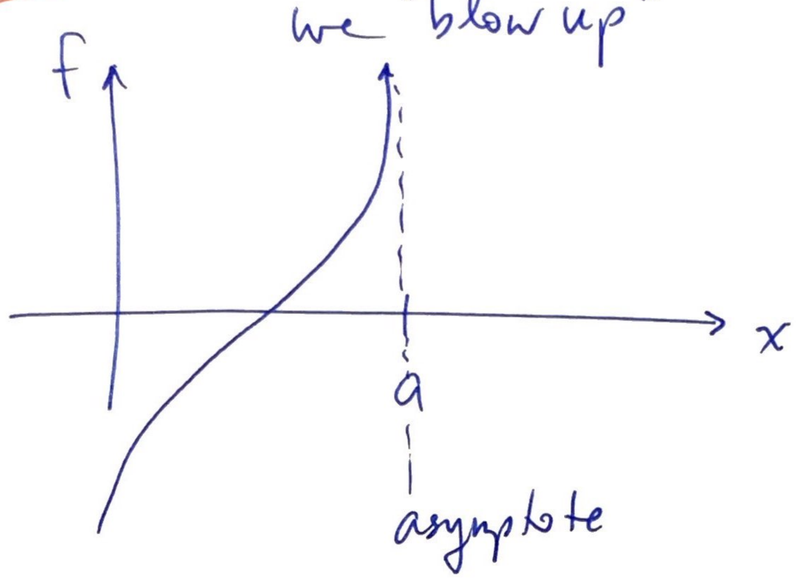
- $\boxed{f(0) = -1}$
- $\boxed{f(3) = 1}$
- $\boxed{\lim_{x \to 0} f(x) = 1}$
- $\boxed{\lim_{x \to 3^-} f(x) = -2}$
- $\boxed{\lim_{x \to 3^+} f(x) = +2}$



III Infinite Limits

Here $\boxed{\lim_{x \to a^-} f(x) = \text{undefined}}$

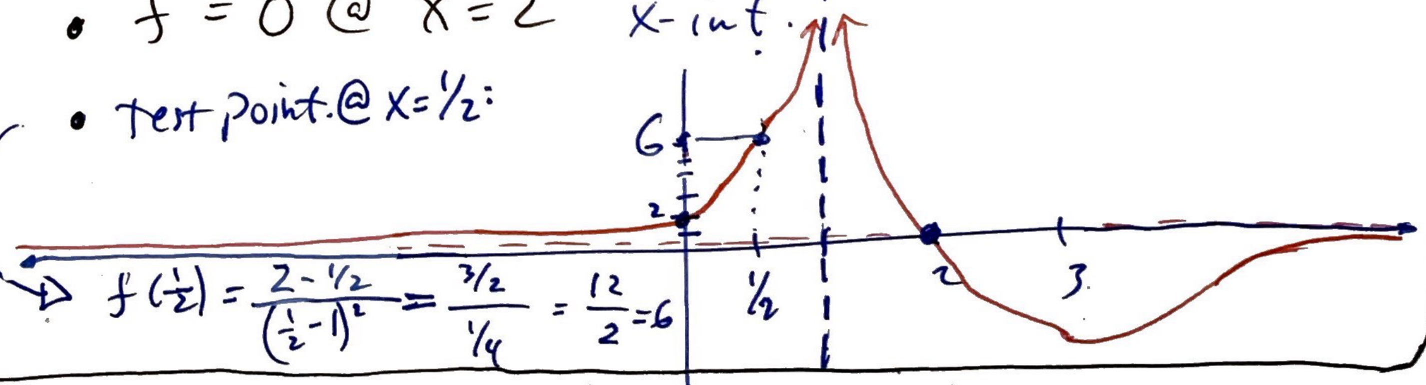
EX



EX Vertical asymptote

Sketch $f(x) = \dfrac{2-x}{(x-1)^2}$ } openstax.com
↳ math → precalculus
((look for sketching s))

- Horizontal Asymptote = 0 since deg on top < deg bot
- Vertical Asymptote = when denom = 0
- f = 2 @ x = 0 Here @ x = 1 {even} power = 2
- f = 0 @ x = 2 x-int.
- test point @ $x = 1/2$:



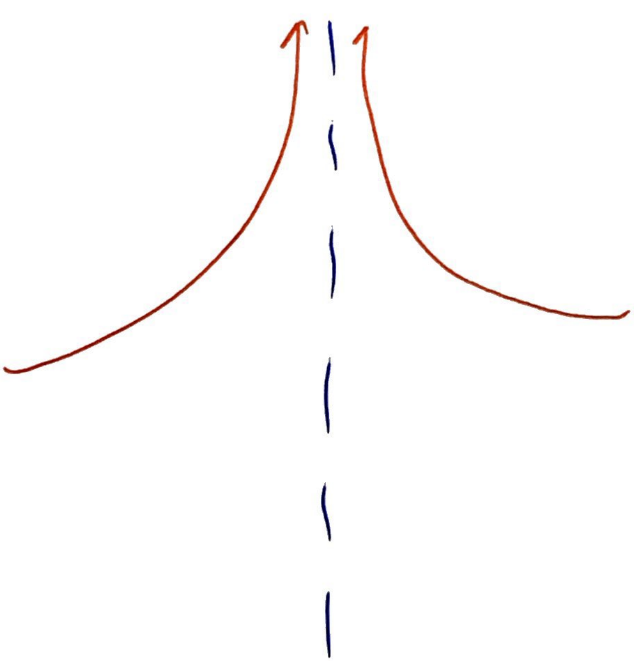
$f(\frac{1}{2}) = \dfrac{2 - 1/2}{(\frac{1}{2}-1)^2} = \dfrac{3/2}{1/4} = \dfrac{12}{2} = 6$

the location of $x=a$ is a vertical asymptote if either case holds:

$$\left.\begin{aligned}&\lim_{x\to a^-} f(x)=\pm\infty\\ &\lim_{x\to a^+} f(x)=\pm\infty\end{aligned}\right\}\quad x=a \text{ even asymptote}$$

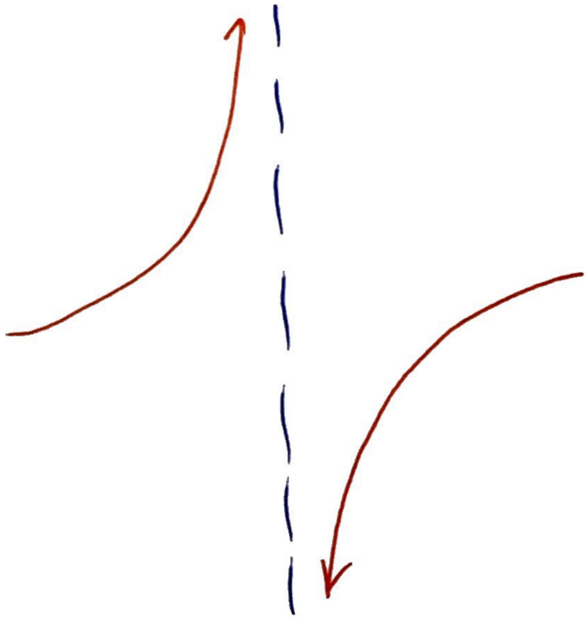
But if

$$\left.\begin{aligned}&\lim_{x\to a^-} f(x)=\pm\infty\\ &\lim_{x\to a^+} f(x)=\mp\infty\end{aligned}\right\}\quad x=a \text{ is an odd asymptote}$$



$n =$ even

$(x-a)^n$



$n =$ odd

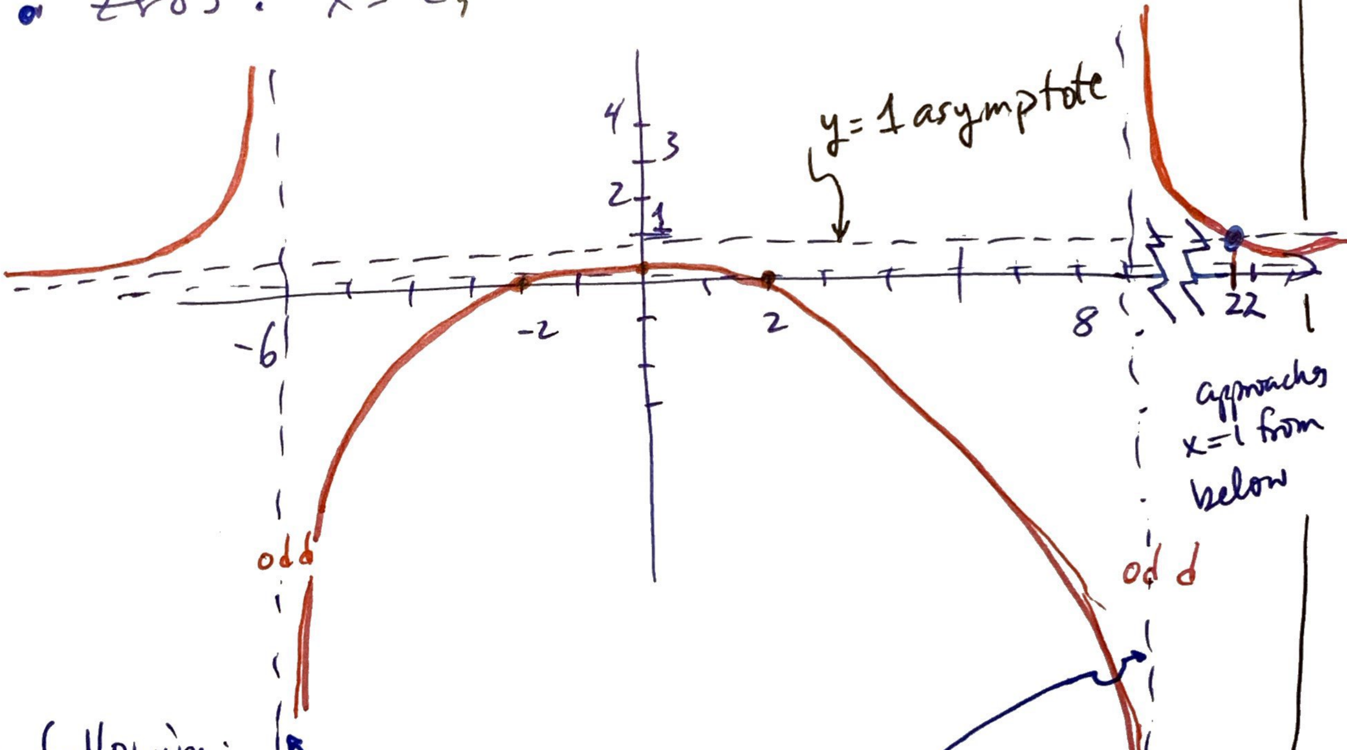
$(x-a)^n$

EX analyze $y = \frac{x^2-4}{x^2-2x-48}$

- factor the denominator and numerator:

$$f(x) = \frac{(x-2)(x+2)}{(x-8)(x+6)}$$

- graph $y=f(x)$:
  - HA: $\frac{x^2}{x^2} = 1 \Rightarrow y = 1$ {when deg top = deg bot}
  - VA: $x = 8$ (odd) power = 1, $x = -6$ (odd) power = 1
  - zeros: $x = -2, 2$



Note the following:

$\lim_{x \to -6^-} f(x) = +\infty$

$\lim_{x \to -6^+} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = 1^+$ from above

$\lim_{x \to 8^-} f(x) = -\infty$

$\lim_{x \to 8^+} f(x) = \infty$

$\lim_{x \to +\infty} f(x) = 1^+$ from above

Does the graph cross $y = 1$?

$$1 = \frac{x^2-4}{x^2-2x-48}$$

$\Rightarrow \cancel{x^2}-2x-48 = \cancel{x^2}-4$

$-2x = -4+48$

$\boxed{x = 22}$

yes! @ ↑