9.7 Inverse Matrices

Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

$2\times 2 \quad 2\times 1 \qquad\qquad 2\times 1$

Then we can write a system of equations like this:

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} \underbrace{=}_{=} \underbrace{\begin{pmatrix} e \\ f \end{pmatrix}}_{\vec{b}} \Rightarrow \boxed{\begin{aligned} ax+by &= e \\ cx+dy &= f \end{aligned}}$$

Ex write $\begin{cases} 3x+y-2z=2 \\ x-2y+z=3 \\ 2x-y-3z=3 \end{cases}$ as a matrix equation

$$\Rightarrow \begin{pmatrix} 3 & 1 & -2 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

let $A = \begin{pmatrix} 3 & 1 & -2 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{pmatrix}$

let $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

let $\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

then the system can be written as

$$A\vec{x} = \vec{b}$$

* Inverse of a matrix:

identity matrix

$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$

Def: If $\mathbb{A}\mathbb{B} = \mathbb{I}$ the $\mathbb{B}$ is the inverse matrix of $\mathbb{A}$. $\mathbb{A}$ is the inverse matrix of $\mathbb{B}$.

$\boxed{\text{EX}}$ Are $\mathbb{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ & $\mathbb{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ inverses of each other?

$$\mathbb{A}\mathbb{B} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} 1\cdot1 + 0\cdot1 & 1\cdot0 + 0\cdot1 \\ \hline -1\cdot1 + 1\cdot1 & -1\cdot0 + 1\cdot1 \end{array}\right] \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$\mathbb{A}$ and $\mathbb{B}$ are inverses of each other.

We write the inverse of $\mathbb{A}$: $\mathbb{A}^{-1}$ $\quad \left\{ \mathbb{A}^{-1} \neq \frac{1}{\mathbb{A}} \right\}$

BTW: $\mathbb{A}\mathbb{I} = \mathbb{A}$

$\mathbb{I}\mathbb{A} = \mathbb{A}$

Then $\mathbb{A}\mathbb{A}^{-1} = \mathbb{I}$ likewise $\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$

Back to our matrix equation.

$$\boxed{\mathbb{A}\vec{x} = \vec{b}}$$

if we can establish $\mathbb{A}^{-1}$ then we can multiply our eqn by $\mathbb{A}^{-1}$, from the left, to get

$$\mathbb{A}^{-1}(\mathbb{A}\vec{x}) = \mathbb{A}^{-1}\vec{b}$$

$$(\mathbb{A}^{-1}\mathbb{A})\vec{x} = \mathbb{A}^{-1}\vec{b}$$

$$\mathbb{I}\vec{x} = \mathbb{A}^{-1}\vec{b}$$

$$\boxed{\vec{x} = \mathbb{A}^{-1}\vec{b}}$$

So to solve a system of equation, "just" find the inverse of $\mathbb{A}$ and use $\vec{x} = \mathbb{A}^{-1}\vec{b}$ to get $\vec{x}$.

$\boxed{\text{EX}}$ If $\mathbb{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ then $\mathbb{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ since $\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$

So use this knowledge to solve

$$\begin{aligned} x \quad &= 4 \\ -x + y &= -3 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} \therefore \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}}$$

Q: How do we find $\mathbb{A}^{-1}$?

$\boxed{2\times 2}$

If $\mathbb{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\boxed{\mathbb{A}^{-1} = \dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - cb}}$

cb

ad

- exchange the main diagonal
- change the signs on the anti-diagonal
- divide by the qnty $ad-cb$

$\boxed{\text{ex:}}$

let $\mathbb{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ then

$$\mathbb{A}^{-1} = \frac{\begin{bmatrix} 1 & -0 \\ -(-1) & 1 \end{bmatrix}}{1\cdot 1 - (-1)(0)}$$

$$\boxed{\mathbb{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$$

BTW $\mathbb{A}^{-1}$ only exists for square matrices

For 3×3 and higher dimensions

(i) Form the augmented matrix $[\mathbb{A} \mid \mathbb{I}]$

(ii) Perform row ops to get this to transform into $[\mathbb{I} \mid \mathbb{B}]$

(iii) $\mathbb{B}$ is $\mathbb{A}^{-1}$ "Armani Method"

$\boxed{\text{EX}}$ Find the inverse of $\begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

(i) Form the augmented matrix $[\mathbb{A} \mid \mathbb{I}]$

$$\left[\begin{array}{ccc|ccc} 3 & -2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

swap to get a pivot up top and "zero" out row 2

(ii) Convert the LHS to $\mathbb{I}$ via row operations

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & -2 & 4 & 1 & 0 & 0 \end{array}\right] *-3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 1 & -3 & 0 \end{array}\right] *2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -3 & 2 \end{array}\right] \text{add}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -3 & 2 \end{array}\right] \div -2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} & -1 \end{array}\right]$$

(iii) extract the inverse

So $\boxed{\mathbb{A}^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} & -1 \end{bmatrix}}$

$\boxed{\text{EX}}$ Use the "Armani Method" (Gaussian Method) to find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

(i) form $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} *-2\,;\,*-1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ *2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ *-3\,;\,*3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} \\ *-2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \div -1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

So $\boxed{A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}}$

Now lets solve a system

$\boxed{\text{Ex}}$ Solve

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2x + 5y + 3z &= -4 \\ x \quad\quad + 8z &= 2 \end{aligned}$$

write in matrix form

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 11 \\ -4 \\ 2 \end{pmatrix}}_{\vec{b}}$$

inverse was found in the previous example

write in inverse form

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 11 \\ -4 \\ 2 \end{pmatrix}}_{\vec{b}}$$

perform the multiplication

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -40(11) + 16\cdot(-4) + 9(2) \\ 13(11) \quad -5(4) \quad -3(2) \\ 5(11) \quad -2(4) \quad -1(2) \end{pmatrix} = \begin{pmatrix} -486 \\ -15 \\ 45 \end{pmatrix}$$

So $\boxed{(x, y, z) = (-486, -15, 45)}$