9.6 Gaussian Elimination

Recall that we added and subtracted equation in order to solve them. The strategy was to eliminate variables and end up with a triangular system (sec. 9.2)

Ex

$$\begin{aligned} 3x - 2y + 5z &= 21 \\ 5x + 4y &= 37 \\ x - 2y - 5z &= 5 \end{aligned}$$

lets get a "1" on the top row

$\Downarrow$

pivot position

$$\begin{aligned} x - 2y - 5z &= 5 \quad *-5,\ *-3 \\ 5x + 4y &= 37 \\ 3x - 2y + 5z &= 21 \end{aligned}$$

a.k.a. $-3R_1 + R_3 \rightarrow R_3$

zero out these 1st

zero this 2nd

$\Downarrow$

$$\begin{aligned} x - 2y - 5z &= 5 \\ 0x + 14y + 25z &= 12 \\ 0x + 4y + 20z &= 6 \end{aligned}$$

$\div 2$ and swap it with R_2

$\Downarrow$

$$\begin{aligned} x - 2y - 5z &= 5 \\ 2y + 10z &= 3 \quad *-7 \\ 14y + 25z &= 12 \end{aligned}$$

$$\begin{aligned} -14y - 70z &= -21 \\ + \quad 14y + 25z &= 12 \end{aligned}$$

$\Downarrow$

$$\left.\begin{aligned} x - 2y - 5z &= 5 \\ 2y + 10z &= 3 \\ -45z &= -9 \end{aligned}\right\}$$ triangular system

results

$\boxed{\text{EX}}$ cont.

Gaussian Elimination

$$\begin{aligned} x - 2y - 5z &= 5 \\ 2y + 10z &= 3 \\ -45z &= -9 \quad \leftarrow \div -9 \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} x - 2y - 5z &= 5 \\ 2y + 10z &= 3 \\ 5z &= 1 \quad \times -2; \end{aligned}$$

Back

choice #1: Substitute

choice #2: continue eliminating

Jordan Elimination

$$\Updownarrow$$

$$\begin{aligned} x - 2y &= 6 \\ 2y &= 1 \\ 5z &= 1 \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} x &= 7 \\ 2y &= 1 \quad \div 2 \\ 5z &= 1 \quad \div 5 \end{aligned}$$

$$\left.\begin{aligned} x &= 7 \\ y &= 1/2 \\ z &= 1/5 \end{aligned}\right\}$$

Answer

$(x, y, z) = \left(7, \frac{1}{2}, \frac{1}{5}\right)$

Triangular system ...

* Augmented Matrix

To save time and writting we drop the $x, y, z, =$ and just deal with the coefficients of the equations.

$\boxed{\text{EX}}$ (previous example)

$$\left.\begin{array}{l} 3x - 2y + 5z = 21 \\ 5x + 4y = 37 \\ x - 2y - 5z = 5 \end{array}\right\} \rightarrow \underbrace{\left(\begin{array}{ccc|c} 3 & -2 & 5 & 21 \\ 5 & 4 & 0 & 37 \\ 1 & -2 & -5 & 5 \end{array}\right)}_{\text{augmented matrix}}$$

the goal is to perform elementary row operations and transform the matrix to a diagonal system

From the previous example, such Row Ops yeilded

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/5 \end{array}\right) \xrightarrow[\text{eqn "space"}]{\text{go back to}} \left.\begin{array}{l} x = 7 \\ y = 1/2 \\ z = 1/5 \end{array}\right\}$$

$\boxed{\text{Practice}}$:

- write in augmented matrix $\left\{\begin{array}{l} 2x - z = 4 \\ 3x + z = -2 \end{array}\right\} \Rightarrow \left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & -2 \end{array}\right)$

- write in equation "space"

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 4 & 0 & 1 & 7 \\ 0 & 3 & -4 & 5 \end{array}\right) \Rightarrow \left.\begin{array}{l} x - 2y + 3z = 0 \\ 4x + z = 7 \\ 3y - 4z = 5 \end{array}\right\}$$

$\boxed{\text{EX}}$ Solve via augmented matrix:

$$\frac{1}{10}x - \frac{1}{5}y + 4z = -\frac{41}{2} \quad *10$$
$$\frac{1}{5}x - 20y + \frac{2}{5}z = -101 \quad *5$$
$$\frac{3}{10}x + 4y - \frac{3}{10}z = 23 \quad *10$$

$$\Downarrow$$

$$x - 2y + 40z = -205$$
$$x - 100y + 2z = -505$$
$$3x + 40y - 3z = 230$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 40 & -205 \\ 1 & -100 & 2 & -505 \\ 3 & 40 & -3 & 230 \end{array}\right) *-1 ; *-3 \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 40 & -205 \\ 0 & -98 & -38 & -300 \\ 0 & 46 & -123 & 845 \end{array}\right) *2$$

$$\left(\begin{array}{cc} 1 & -2 \\ & \\ & \end{array}\right)$$

etc. Sorry not workable in our time frame...

EX Solve via augmented matrix

$$\left.\begin{aligned} 3x + y - 2z &= -7 \\ 2x + 2y + z &= 9 \\ -x - y + 3z &= 6 \end{aligned}\right\} \quad \left(\begin{array}{ccc|c} 3 & 1 & -2 & -7 \\ 2 & 2 & 1 & 9 \\ -1 & -1 & 3 & 6 \end{array}\right)$$

make a nice pivot

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & 6 \\ 2 & 2 & 1 & 9 \\ 3 & 1 & -2 & -7 \end{array}\right) {*2 ; *3} \quad \left(\begin{array}{ccc|c} -1 & -1 & 3 & 6 \\ 0 & 0 & 7 & 21 \\ 0 & -2 & 7 & 11 \end{array}\right)$$

make a nice pivot

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & 6 \\ 0 & -2 & 7 & 11 \\ 0 & 0 & 7 & 21 \end{array}\right) \div 7 \quad \left(\begin{array}{ccc|c} -1 & -1 & 3 & 6 \\ 0 & -2 & 7 & 11 \\ 0 & 0 & 1 & 3 \end{array}\right) {*-7 ; *-3}$$

already done!!

$$\left(\begin{array}{ccc|c} -1 & -1 & 0 & -3 \\ 0 & -2 & 0 & -10 \\ 0 & 0 & 1 & 3 \end{array}\right) \div -2 \quad \left(\begin{array}{ccc|c} -1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right) +$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right) {*-1} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\Rightarrow \quad \begin{aligned} x + 0y + 0z &= -2 \\ 0x + y + 0z &= 5 \\ 0x + 0y + z &= 3 \end{aligned} \qquad \begin{aligned} x &= -2 \\ y &= 5 \\ z &= 3 \end{aligned}$$

$$\boxed{(x, y, z) = (-2, 5, 3)}$$