$\boxed{9.4}$ Partial Fractions

In calculus $\int \frac{dx}{x^2-4}$

one way to simplify $\frac{1}{x^2-4}$ is to break this more complicated integrand into two simple fraction

(0) factor

(i) State the forms

Observe: $$\frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

Alg II: $\frac{-1}{4(x+2)} + \frac{1}{4(x-2)}$

Combine these

(ii) Common Denominators

$$\boxed{\frac{1}{x^2-4} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}}$$

$$\frac{A}{x+2}\left(\frac{x-2}{x-2}\right)+\frac{B}{x-2}\left(\frac{x+2}{x+2}\right) = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$

$$\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$$

(iii) Numerators are equal on the LHS and RHS

$$\boxed{1 = A(x-2)+B(x+2)}$$

$$1 = Ax+Bx-2A+2B$$

(iv) separate powers

$$0x^1+1x^0 = (A+B)x^1+(2B-2A)x^0$$

In Linear Algebra we show that if $\alpha x^2+\beta x+\gamma = sx^2+tx+u$ that $\alpha = s$, $\beta = t$ and $\gamma = u$

$\Rightarrow$

(v) 2 eqns & 2 unknowns

	LHS		RHS
$\Rightarrow x^1$:	$0x$	$=$	$(A+B)x$
x^0:	1	$=$	$2B-2A$

Lets solve this system:

$$\begin{cases} A + B = 0 & *2 \\ -2A + 2B = 1 \end{cases} \Rightarrow \begin{cases} 2A + 2B = 0 \\ -2A + 2B = 1 \end{cases}$$

$\oplus$

$$4B = 1 \qquad \boxed{B = 1/4}$$

But since $A+B=0$ $\boxed{A \text{ is } -1/4}$

So $$\frac{1}{x^2-4} = \frac{-1/4}{x+2} + \frac{1/4}{x-2}$$

(vi) Complete the decomposition

i.e. $$\boxed{\frac{1}{x^2-4} = -\frac{1}{4(x+2)} + \frac{1}{4(x-2)}}$$

Partial Fraction Decomposition

$\boxed{\text{EX}}$ $$\frac{4x+3}{\underset{3,5}{x^2+8x+15}} \overset{(i)}{=} \frac{4x+3}{(x+3)(x+5)} \overset{(ii)}{=} \frac{A}{x+3} + \frac{B}{x+5}$$

$$\Rightarrow \frac{4x+3}{(x+3)(x+5)} = \frac{A(x+5)+B(x+3)}{(x+3)(x+5)}$$

(iii) • Numerators: $4x+3 = (A+B)x + (5A+3B)$

(v) solved

(iv) • Equate Powers: $\begin{cases} x^1: 4 = A+B & *-5 \\ x^0: 3 = 5A+3B \end{cases}$ $\begin{cases} 4 = A+B \\ -17 = -2B \end{cases}$ $\boxed{A = -9/2}$ $\boxed{B = 17/2}$

(vi) • Final Form $$\boxed{\frac{4x+3}{x^2+8x+15} = -\frac{9}{2(x+3)} + \frac{17}{2(x+5)}}$$

EX

* If we have

$$\frac{4x^2-8x+7}{(x-2)(x+1)(x+3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$= \frac{A(x+1)(x+3) + B(x-2)(x+3) + C(x-2)(x+1)}{(x-2)(x+1)(x+3)}$$

$$\boxed{4x^2 - 8x + 7 = Ax^2 + 4Ax + 3A + Bx^2 + Bx - 6B + Cx^2 - Cx - 2C}$$

$$\left\{\begin{array}{l} x^2 : \ 4 = A + B + C \\ x^1 : \ -8 = 4A + B - C \\ x^0 : \ 7 = 3A - 6B - 2C \end{array}\right\} \ 3\times 3 \ \text{(see sec 9.2)}$$

use matrixcalc.org $\Rightarrow A = \frac{7}{15},\ B = \frac{-19}{6},\ C = \frac{67}{10}$

$$\Rightarrow \boxed{\frac{4x^2-8x+7}{(x-2)(x+1)(x+3)} = \frac{7}{15(x-2)} - \frac{19}{6(x+1)} + \frac{67}{10(x+3)}}$$

to be continued...

Completes Linear terms, non-repeated

* Repeating Linear fractions

Ex: $$\frac{-5x-19}{(x+4)^2} = \frac{A}{x+4} + \frac{B}{x+4} \ ? \quad \text{No!}$$

(i) form
$$-\frac{5x+19}{(x+4)^2} = \frac{A}{x+4} + \frac{B}{(x+4)^2}$$

Top is one degree less than the factor in the bottom.

(ii) common denom
$$-\frac{5x+19}{(x+4)^2} = \frac{A\,(x+4)}{(x+4)(x+4)} + \frac{B}{(x+4)^2}$$

$=1$

(iii) equate numerators
$$\boxed{-5x-19 = A(x+4)+B}$$

(iv) match powers:

linear x^1: $-5 = A$

constant x^0: $-19 = 4A+B$

(v) Solve:
$$A=-5, \quad B=-19-4(-5)=1$$

(vi) final form:
$$\boxed{\frac{-5x-19}{(x+4)^2} = \frac{-5}{x+4} + \frac{1}{(x+4)^2}}$$

$\boxed{\text{EX}}$

$$\frac{2x^2+4x-1}{(x+2)^3} = \frac{A}{(x+2)^1} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

$$\frac{2x^2+4x-1}{(x+2)^3} = \frac{A(x+2)^2}{(x+2)^3} + \frac{B(x+2)}{(x+2)^3} + \frac{C}{(x+2)^3}$$

$$2x^2+4x^1-1x^0 = Ax^2+4Ax+4A+Bx+2B+C$$

x^2: $\boxed{2 = A}$

x^1: $4 = 4A+B \rightarrow \boxed{B=-4}$

x^0: $-1 = 4A+2B+C$

$-1 = 4(2)+2(-4)+C \rightarrow \boxed{C=-1}$

final form

$$\boxed{\frac{2x^2+4x-1}{(x+2)^3} = \frac{2}{x+2} - \frac{4}{(x+2)^2} - \frac{1}{(x+2)^3}}$$

$\boxed{\text{EX}}$ $$\frac{x^3-5x^2+12x+144}{x^2(x^2+12x+36)}$$

$$\frac{x^3-5x^2+12x+144}{x^2(x+6)^2} = \boxed{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+6}+\frac{D}{(x+6)^2}}$$

$\boxed{\text{EX}}$ $$\left.\frac{x^4-3x^2+x-1}{x^4+12x^3+36x^2}\right\}$$ Improper rational expression

So we need to Long divide 1st

$$\begin{array}{r} 1 \\ x^4+12x^3+36x^2+0x+0 \;\big)\; x^4+0x^3-3x^2+x-1 \\ -(x^4+12x^3+36x^2+0x+0) \\ \hline -12x^3-39x^2+x-1 \end{array}$$

so

$$\frac{x^4-3x^2+x-1}{x^4+12x^3+36x^2} = 1 + \underbrace{\frac{-12x^3-39x^2+x-1}{x^4+12x^3+36x^2}}_{\text{proper rational exp.}}$$

$$\frac{x^4-3x^2+x-1}{x^4+12x^3+36x^2} = 1 + \frac{-12x^3-39x^2+x-1}{x^2(x+6)^2}$$

(i) partial fraction decomposition on the 2^{nd} term only.

$$\frac{-12x^3-39x^2+x-1}{x^2(x+6)^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+6}+\frac{D}{(x+6)^2}$$

$\Rightarrow$ solve this problem

(ii)

$$\boxed{\frac{x^4-3x^2+x-1}{x^4+12x^3+36x^2} = 1+\frac{\#}{x}+\frac{\#}{x^2}+\frac{\#}{x+6}+\frac{\#}{(x+6)^2}}$$

* Irreducible Quadratic Terms

$x^2+2x+7 = (x \quad ?)(x \quad ?)$? "prime" can't factor

Fund. Thm of Algebra

Polynomials can factor $= (x-r_1)(x-r_2)\cdots(x-r_k)\cdot$
$(x^2+b_1x+c_1)(x^2+b_2x+c_2)\cdots$

Another version says we can factor those irreducable quadratic factor into conjugate pairs.

So $x^2+b_1x+c_1 = \underbrace{(x-(\alpha+i\beta))(x-(\alpha-i\beta))}_{\text{linear complex conjugate terms.}}$

We will accept the real irreducible (prime) quadratic terms.

The form for these involves a linear numerator ($mx+b$)

EX $\boxed{\dfrac{4x^2+9x+23}{(x-1)(x^2+6x+11)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+6x+11}}$

Hint: the numerators are one degree less than the denominators on the RHS.

EX * repeating Quadratic Factor

$$\boxed{\frac{x^3+2x^2+4x}{(x^2+2x+9)^2} = \frac{Ax+B}{x^2+2x+9} + \frac{Cx+D}{(x^2+2x+9)^2}}$$

EX

$$\frac{x^2-3x+1}{(x-6)^2(x^2+6)^3} = \underbrace{\frac{A}{x-6} + \frac{B}{(x-6)^2}} + \underbrace{\frac{Cx+D}{x^2+6} + \frac{Ex+F}{(x^2+6)^2} + \frac{Gx+H}{(x^2+6)^3}}$$

EX #49

$$\frac{x^2+25}{(x^2+3x+25)^2} = \frac{Ax+B}{x^2+3x+25}\cdot\left(\frac{x^2+3x+25}{x^2+3x+25}\right) + \frac{Cx+D}{(x^2+3x+25)^2}$$

$$\frac{x^2+25}{(x^2+3x+25)^2} = \frac{(Ax+B)(x^2+3x+25)}{(x^2+3x+25)^2} + \frac{Cx+D}{(x^2+3x+25)^2}$$

numerators:

$$x^2+25 = \underbrace{Ax^3+3Ax^2+25Ax+Bx^2+3Bx+25B} + Cx+D$$

Solve:

$x^3: 0 = A \longrightarrow A=0$

$x^2: 1 = 3A+B \longrightarrow B=1$

$x^1: 0 = 25A+3B+C \longrightarrow C=-3$

$x^0: 25 = 25B+D \longrightarrow D=0$

final form

$$\boxed{\frac{x^2+25}{(x^2+3x+25)^2} = \frac{1}{x^2+3x+25} + \frac{-3x}{(x^2+3x+25)^2}}$$

$$= \frac{1}{(x+a)^2+b}$$