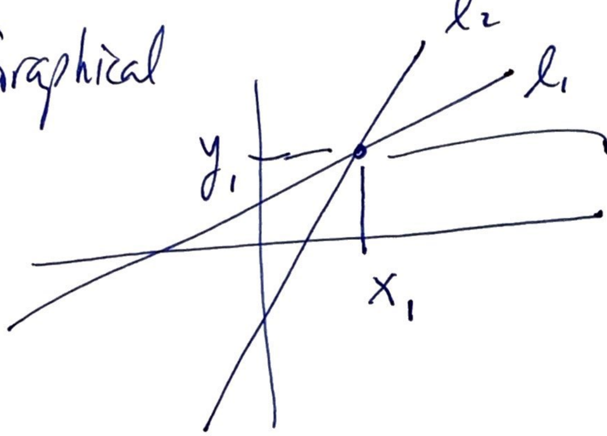
Chapter 9 : Systems of equations

9.1 (review only, No HW) 2-Dim linear Systems

$$\begin{cases} 3x - 2y = 4 : \ell_1 \\ x + 5y = -1 : \ell_2 \end{cases}$$

① Graphical



(x_1, y_1) would simultaneously satisfy <u>both</u> eqns

② Substitution $\quad 3x - 2y = 4$
$\qquad\qquad\qquad\quad x + 5y = -1$

(i) pick the simplest eqn, $x + 5y = -1$

(ii) solve for the cleanest variable (x) here

$$x = -1 - 5y$$

(iii) substitute this into the non-choosen eqn.

$$3(-1-5y) - 2y = 4$$

(iv) Solve $\quad -15y - 2y = 4 + 3 \Rightarrow -17y = 7 \Rightarrow y = -\frac{7}{17}$

(v) insert this into the simplest eqn solved for the cleanest var. $X = -1 - 5\left(-\frac{7}{17}\right) = \frac{18}{17}$ $\boxed{\left(\frac{18}{17}, -\frac{7}{17}\right)}$

3 Elimination (Gaussian & Gauss-Jordan)

$\boxed{Ex}$

$$\left.\begin{array}{l} 3x-2y=4 \\ x+5y=-1 \quad * -3 \end{array}\right\} \Rightarrow \begin{array}{r} 3x-2y=4 \\ \oplus\ -3x-15y=3 \\ \hline 0x-17y=7 \end{array}$$

$$\Rightarrow \boxed{y=-\frac{7}{17}}$$

Then we back substitute this into the system to get x: (Gaussian)

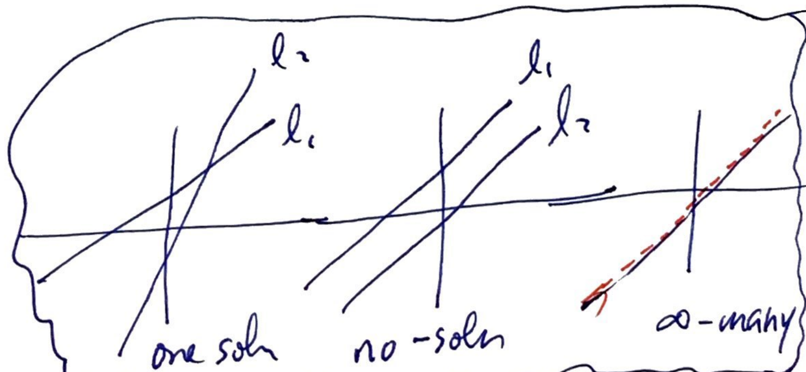
Use: $x+5y=-1$

$$x+5\left(-\frac{7}{17}\right)=-1$$

$$x=-1+\frac{35}{17} \Rightarrow \boxed{x=\frac{18}{17}}$$

Future methods:

- Gauss-Jordan
- Inverse Matrix
- Cramer's Rule



* If we get "non-sense" contradictions, then the lines are $\parallel$ and there is no-soln.

* If we see one eqn is a fraction of the other, we have $\boxed{\infty\text{-many solns}}$

9.2 3-dimensional systems (Linear)

So now we add another dimension

Ex Show that $(4, 2, -6)$ is the solution to the intersections of these three plane:

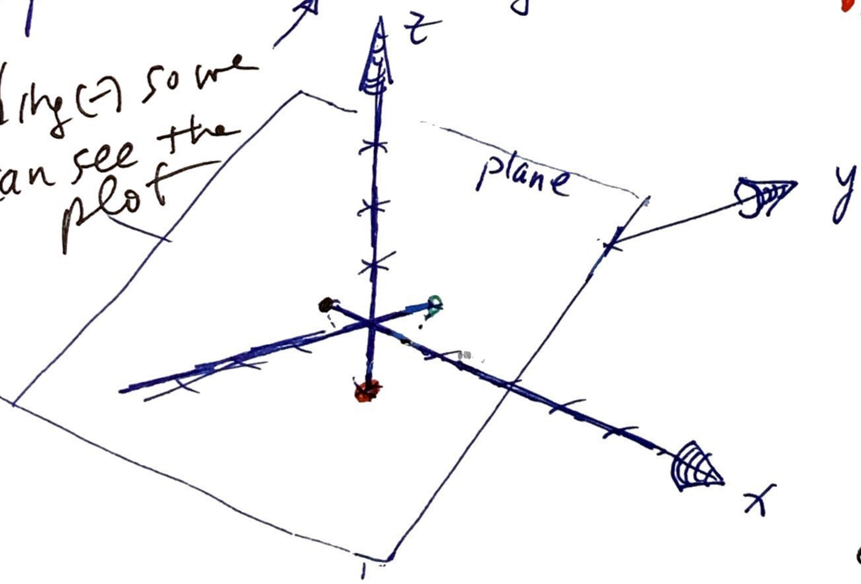
$$\left.\begin{aligned} 6x - 7y + z &= 2 \\ -x - y + 3z &= 4 \\ 2x + y - z &= 1 \end{aligned}\right\} \quad \text{3 eqns and 3 unknown } (x, y, z)$$

Here these are <u>linear since</u> the powers on x, y & z are <u>all</u> 1. $\{y = mx + b\}$

But in 3-D these are planes

- Graph one: $-2x + y - z = 1$

adding (-) so we can see the plot

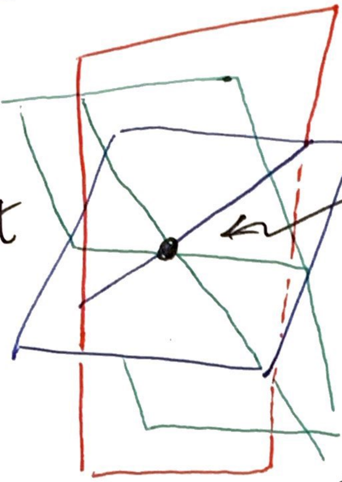


"cover up method"

- let x, y be zero then $z = -1$ $(0, 0, -1)$
- let x, z be zero $y = 1$ $(0, 1, 0)$
- let $y, z = 0$ $x = -1/2$ $(-\frac{1}{2}, 0, 0)$

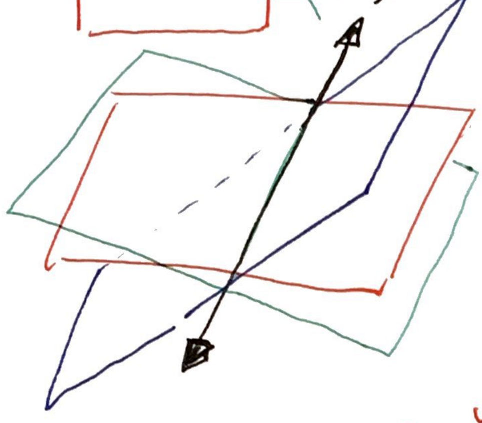
* diffrent configurations

point

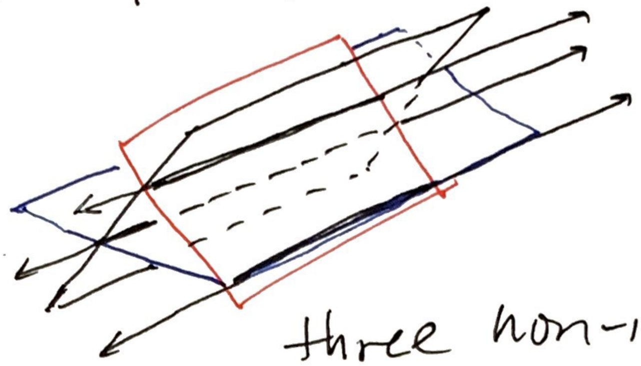


(x, y, z) solution is common to all three eqns. (planes)

line



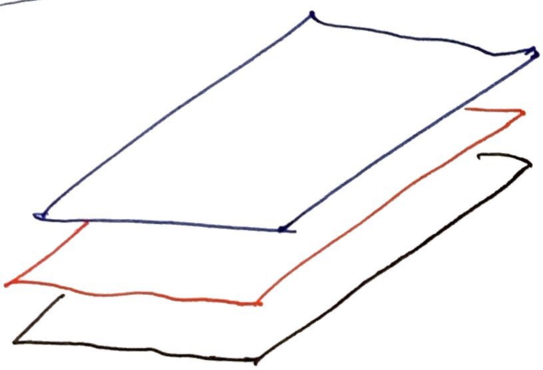
"consistant solns"



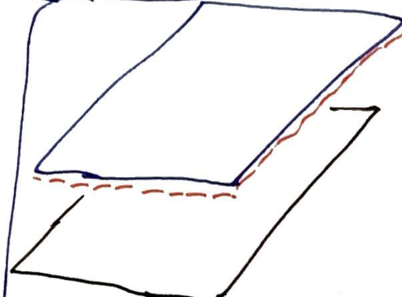
three non-intersecting lines

"No Solution"

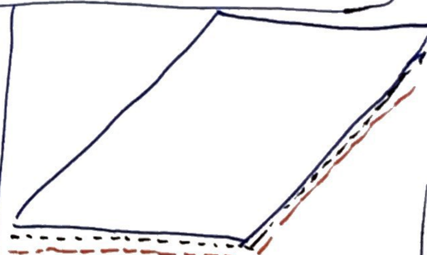
"Inconsistant system"



no soln.



no soln



- all three are "coplanar"

∞-many solns

* Lets solve a 3-d system by substitution 5

EX #12

$$5x-2y+3z=20$$
$$2x-4y-3z=-9$$
$$x+6y-8z=21$$

3 eqns & 3 unknown

- Solve the bottom eqn for x

$$\boxed{x=21-6y+8z}$$

$x=21-6(4)+8(1$

$\boxed{x=5}$

- substitute

$$2(21-6y+8z)-4y-3z=-9$$
$$5(21-6y+8z)-2y+3z=20$$

- clean up

$$42-12y+16z-4y-3z=-9$$
$$105-30y+40z-2y+3z=20$$

- clean up

$$-16y+13z=-51$$
$$-32y+43z=-85$$

2 eqns & 2 unknowns

- prepare the top eqn

$$y=\frac{-51}{-16}-\frac{13}{-16}z \Rightarrow \boxed{y=\frac{13}{16}z+\frac{51}{16}} \Rightarrow \boxed{y=4}$$

- substitute

$$-32\left[\frac{13}{16}z+\frac{51}{16}\right]+43z=-85$$

$$-26z-102+43z=-85$$

$$17z=17$$

1 eqn & 1 unknown

$$\boxed{z=1}$$

$$\begin{array}{r} 102 \\ -85 \\ \hline 17 \end{array}$$

- Ans: $\boxed{(x,y,z)=(5,4,1)}$

Solve by Gaussian Elimination

Here we add multiples of one eqn to another eqn.

∘ clean up

EX #20

$4x + 6y - 2z = 8$ ← all even #'s so $\div 2$

$6x + 9y - 3z = 12$ ← $\div 3$

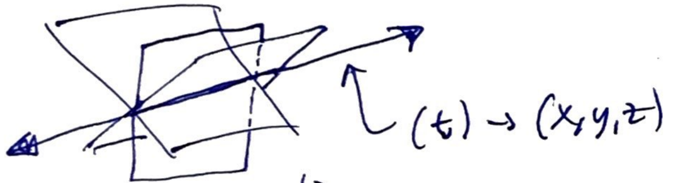
$-2x - 3y + z = -4$ ← $\div -1$

∘ observation

$$\left.\begin{aligned} 2x + 3y - z &= 4 \\ 2x + 3y - z &= 4 \\ 2x + 3y - z &= 4 \end{aligned}\right\}$$ all the same plane

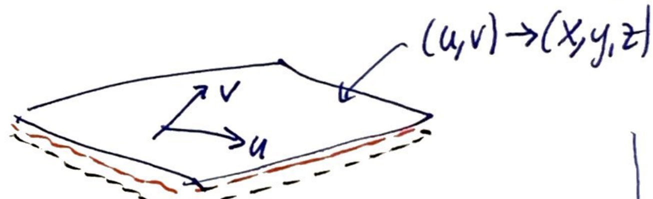
∞ 2-dim solutions

Knowledge break



one dim solns

∞ many constrained to a line



all 3 coplanar

∞ many 2-dim solns. on the plane.

∘ parameterize solution

So here just take one $2x+3y-z=4$

and solve for x: $x = 2 + \frac{1}{2}z - \frac{3}{2}y$

• let $z = s$ and let $y = t$, then

$x = 2 - \frac{1}{2}s - \frac{3}{2}t$

• all together

$\boxed{(x, y, z) = \left(-\frac{1}{2}s - \frac{3}{2}t + 2,\ t,\ s\right)}$ for any s & $t \in \mathbb{R}$

parametric form of the solution

EX Simplify then use Gaussian Elimination to solve.

$$\frac{1}{2}x - \frac{1}{5}y + \frac{2}{5}z = -\frac{13}{10}$$
$$\frac{1}{4}x - \frac{2}{5}y - \frac{1}{5}z = -\frac{7}{20}$$
$$-\frac{1}{2}x - \frac{3}{4}y - \frac{1}{2}z = -\frac{5}{4}$$

• clear fractions →

$$\boxed{\begin{aligned} 5x - 2y + 4z &= -13 \\ 5x - 8y - 4z &= -7 \\ 2x + 3y + 2z &= 5 \end{aligned}}$$

• normal Gaussian ~~proceedure~~

$5x - 2y + 4z = -13 \quad * -1$
$5x - 8y - 4z = -7$
$2x + 3y + 2z = 5$

multiply row 1 by -1 then add that to row 2 and replace row 2.

• To avoid fractions $-1 \cdot R_1 + R_2 \rightarrow R_2$

$5x - 2y + 4z = -13 \quad * -2$
$0x - 6y - 8z = 6$
$2x + 3y + 2z = 5 \quad * 5$

$-10x + 4y - 8z = 26$
$0x - 6y - 8z = 6$
$10x + 15y + 10z = 25$

$5x - 2y + 4z = -13$
$-6y - 8z = 6 \quad \div 2$
$19y + 2z = 51 \quad * 2$

focus on these

$5x - 2y + 4z = -13$
$-3y - 4z = 3$
$38y + 4z = 102$

$5x - 2y + 4z = -13$
$-3y - 4z = 3$
$35y = 105 \rightarrow \boxed{y = 3}$

we have finished the Gaussian Elim. and now we back substitite.

$-3(3) - 4z = 3 \rightarrow \boxed{z = -3}$

$5x - 2(3) + 4(-3) = -13$
$5x = -13 + 12 + 6$
$\boxed{x = 1}$

So
$$\boxed{(x, y, z) = (1, 3, -3)}$$

Another example

EX #44

$$\begin{aligned} 0.3x + 0.3y + 0.5z &= 0.6 \\ 0.4x + 0.4y + 0.4z &= 1.8 \\ 0.4x + 0.2y + 0.1z &= 1.6 \end{aligned}$$

* clear fractions * $*10$

$\Downarrow$

* make a one *

$$\begin{aligned} 3x + 3y + 5z &= 6 \quad + \\ 4x + 4y + 4z &= 18 \quad *-1 \; ; \div 2 \\ 4x + 2y + z &= 16 \end{aligned}$$

$\Downarrow$

* normal strategy *

$$\begin{aligned} -x - y + z &= -12 \quad *2 \; ; *4 \\ 2x + 2y + 2z &= 9 \\ 4x + 2y + z &= 16 \end{aligned}$$

$\Downarrow$

* start back substitution *

$$\begin{aligned} -x - y + z &= -12 \\ 4z &= -15 \Rightarrow \boxed{z = \tfrac{-15}{4}} \\ -2y + 5z &= -32 \end{aligned}$$

(i) (ii)

$$-2y + 5\left(-\frac{15}{4}\right) = -32$$

$$y = \frac{-32}{-2} + \frac{75}{-8} = \frac{128}{8} - \frac{75}{8} = \boxed{\frac{53}{8} = y}$$

$$-x - \left(\frac{53}{8}\right) + \left(-\frac{15}{4}\right) = -12$$

$$-x = -12 + \frac{53}{8} + \frac{15}{4} \cdot \frac{2}{2}$$

$$x = 12 - \frac{53}{8} - \frac{30}{8}$$

$$x = \frac{96}{8} - \frac{83}{8}$$

$$x = \frac{13}{8}$$

$$\boxed{(x, y, z) = \left(\frac{13}{8}, \frac{53}{8}, -\frac{30}{8}\right)}$$

9. 2 is done