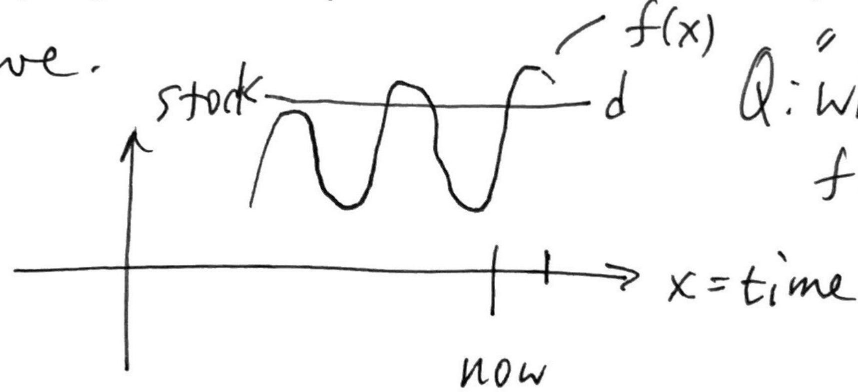
$\boxed{3.6}$ Zeros of a Polynomial Function

Why: We model many phenomena using polynomials. And we seek to know when the hit a certail value.



Q: "when" is $f(x) = d$?

but $f(x) = d \Rightarrow \underbrace{f(x) - d}_{g(x) = \text{polynomial itself}} = 0$

Objective: $f(x) = 0$ @ $x = a, b, c, \ldots$

zeros of the polynomial

$\boxed{\text{Q: How do we find these zeros?}}$

How: We will use theorems to estimate various facts about the quantity of zeros, the location of zeros and the candidate values of zeros.

*Tools 1st

- Recall $\frac{13}{2} = 6 \text{ r } 1 \Rightarrow \boxed{6 + \frac{1}{2}}$

likewise $\frac{f(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}$ } • long divide • if $q(x)$ is linear we can use synthetic division

- clear the fractions

$$\left(\frac{f}{q} = g + \frac{r}{q}\right) * q$$

$$\Rightarrow f(x) = q(x)\,g(x) + r(x)$$

- assume $q(x)$ = a linear binomial $(x-d)$

then $\boxed{f(x) = (x-d)\,g(x) + r}$

{use synthetic division to determine $g(x)$ & $r(x)$

- Note #1 : If $r=0$ we have factored, partially, $f(x) = (x-d)\,g(x)$

- Note #2 : If $x=d$ then $f(d) = (d-d)\,g(d) + r$

so $f(d) = r$ Remainder Theorem

- Note #3 if $f(d) = r$ but $r=0$ we know "d" is a zero of the polynomial.

BTW: we now know three ways to determine the remainder, r, when $x-d$ is divided into $f(x)$

(i) plug & chug : $f(d) = a_n d^n + a_{n-1} d^{n-1} \cdots + a_0$

(ii) long divide : $x-d \overline{\big) a_n x^n + a_{n-1} x^{n-1} + \cdots a_0}$

(iii) synthetic div : $\underline{d} \big|\ a_n \ \ a_{n-1} \ \cdots a_1 \ \ a_0$

reduced polyn $\rightarrow$ $\overline{\qquad g(x) \qquad}$ r

* assuming $x=d$ is a zero then $\underline{f(x) = (x-d)g(x)}$

then we seek zeros for $g(x)$ instead of $f(x)$ b/c $g(x)$ will be one degree smaller than $f(x)$.

$\Rightarrow$ $g(x) = (x-e)h(x)$

$h(x) = (x-f)\, j(x)$ etc.

So that

$$\boxed{f(x) = c(x-d)(x-e)(x-f)\cdots(x-p)}$$

and we have fully factored $f(x)$ and we know all of its zeros: $d, e, f, \ldots, p$

⊛ Quantity

So where do we start to look?

How about Quantity?

Q: How many zeros does a polynomial have?

A: If the degree of a polynomial is "n", then we have "n" zeros   Fund. Thm of Algebra

Q: How many of these zeros are positive? (>0)

A: Use the De Cartes rule of signs on $f(x)$.

Thm Let $f(x)$ be a polynomial in std. form.

(+): The number of (+) zeros is the number of sign changes across the terms of the polynomial or an even number less than that number.

(−): The number of (−) zeros is the number of sign changes across the terms of the polynomial $f(-x)$, less an even number.

EX. $f(x) = 4x^4 - 2x^2 + 5x + 1$ 2 or 0 (+)

$f(-x) = 4x^4 - 2x^2 - 5x + 1$ 2 or 0 (−)

⊛ Candidates to try

Thm "Rational zeros Theorem"

let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

If $\frac{p}{q}$, in its lowest form, is a rational zero then "p" is a factor of a_0 & "q" is a factor of a_n

* There could still be irrational zeros or complex number zeros. But if there are rational number zeros, it follow this thm.

* Our pool of candidates will then be written by the following

$$\pm \frac{\text{Factors of } a_0}{\text{Factors of } a_n}$$

$\boxed{\text{Ex}}$ for $f(x) = 4x^4 - 2x^2 + 5x + 3$

if this polynomial has rational zeros then the belong to the set at the bottom of this page:

$$\pm \frac{3, 1}{4, 2, 1} = \pm\left(\frac{3}{4}, \frac{3}{2}, \frac{3}{1}, \frac{1}{4}, \frac{1}{2}, \frac{1}{1}\right)$$

$$\boxed{\left\{-3, -\frac{3}{2}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 3\right\}}$$

Bounds on the zeros

* We need not examine candidates larger than "b" if the synthetic divison of "b" into f yeilds a bottom line consisting of all (+) numbers

b is an upper bound.

* we need not examine numbers lower than "l" if the synthetic division of l into f yeilds a bottom line of alternating signs.

EX

$$\begin{array}{r|rrrr} 2 & 3 & 11 & -6 & -8 \\ & & 6 & 34 & 56 \\ \hline & 3 & 17 & 28 & 48 \end{array}$$

← all (+) numbers so 2 is an upper bound.

EX

$$\begin{array}{r|rrrl} -8 & 3 & 11 & -6 & -8 \\ & & -24 & 104 & \text{big } (-) \# \\ \hline & 3 & -13 & 98 & \text{another } (-) \# \\ & + & - & + & - \end{array}$$

← altenating numbers so we need not examin numbers less than -8

"Desperately Seeking ~~Susan~~ zeros"

Putting everything together: $P(x) = 3x^3 + 11x^2 - 6x - 8$

1. Degree: $n = 3$

2. number (+): $3x^3 \underbrace{+11x^2 -}_{1} 6x - 8$ so **1 (+)**

3. number (−): $\underbrace{-3x^3 +}_{1} 11x^2 \underbrace{+6x -}_{2} 8$ so **2 or 0** (−)

4. Rational Zero Candidates:

$$\frac{p}{q} = \pm\frac{\text{factors of } 8}{\text{factors of } 3} = \pm\frac{1,2,4,8}{1,3} = \frac{1}{1}, \frac{2}{1}, \frac{4}{1}, \frac{8}{1} \text{ or}$$

$$\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}$$

• form our set

$$\boxed{\left\{ \cancel{-8}, -4, -\frac{8}{3}, \cancel{-2}, -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \cancel{2}, \cancel{\frac{8}{3}}, \cancel{4}, \cancel{8} \right\}}$$

5. Upper Bound: • try 2:

$$\begin{array}{r|rrrr} 2 & 3 & 11 & -6 & -8 \\ & & 6 & 34 & 56 \\ \hline & 3 & 17 & 28 & 48 \end{array}$$

$\neq 0$ yes, **2 is an UB**

6. Lower Bound: • try −2

$$\begin{array}{r|rrrr} -2 & 3 & 11 & -6 & -8 \\ & & -6 & -10 & +32 \\ \hline & 3 & 5 & -16 & 24 \end{array}$$

$\neq 0$ No L.B yet.

• try −8

$$\begin{array}{r|rrrr} -8 & 3 & 11 & -6 & -8 \\ & & -24 & 104 & -784 \\ \hline & 3 & -13 & 98 & -792 \end{array}$$

$\neq 0$

yes −8 is a LB

7. Start processing the remaining cadidates

Hint : start with 1

$$\begin{array}{r|rrrr} \underline{1} & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array}$$

reduced polynomial

x = 1 is a zero of P(x)

$$\text{now } P(x) = (x-\underline{1})\underbrace{(3x^2+14x+8)}$$

8. now we need the zeros of $3x^2+14x+8$

choices:
- restart the whole process on this polynomial
- Continue hunting on the original
- or, since this reduced polynomial is a quadratic, we can use Quad. Form.
- or otherwise factor by trial-n-error.

$$3x^2+14x+8$$

Trial-n-error :

$$\begin{array}{ll} 3,1 & 2,4 \quad \text{yes} \\ & 1,8 \end{array}$$

$$(3x+2)(1x+4)$$

9. Final factored form :

$$\boxed{P(x) = (x-1)(3x+2)(x+4)}$$

zeros are

$$\boxed{-4, -\tfrac{2}{3}, 1}$$

* Graphing

$\boxed{\text{EX}}$ $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$

- $n = 4$
- $\#(+) = 1$
- $\#(-) = 3$ or 1

} so @ least one (+) & one (−).

- Candidates : $\pm(75, 25, 15, 5, 3, 1)$

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 22 & 50 & -75 \\ & & 1 & 3 & 25 & 75 \\ \hline & 1 & 3 & 25 & 75 & \boxed{0} \end{array}$$

← and an U.B.

$x = 1$ is my only (+) zero.

- $x^3 + 3x^2 + 25x + 75 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 25 & 75 \\ & & -1 & -2 & -23 \\ \hline & 1 & 2 & 23 & +\# \end{array}$$

$$\begin{array}{r|rrrr} -5 & 1 & 3 & 25 & 75 \\ & & -5 & 10 & -125 \\ \hline & 1 & -2 & 35 & -\# \end{array}$$

← lower bound

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 25 & 75 \\ & & -3 & 0 & -75 \\ \hline & 1 & 0 & 25 & \boxed{0} \end{array}$$

$x = -3$ is my only (−) zero

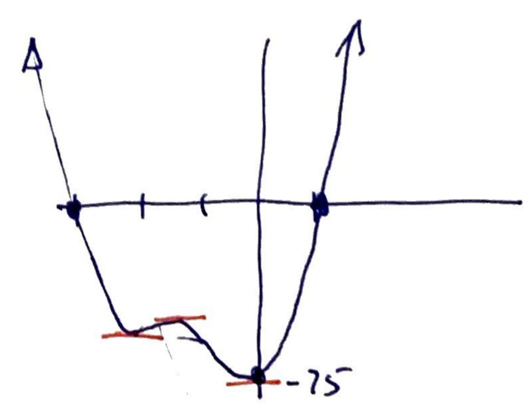
- $x^2 + 25 = 0$

$x^2 = -25$

$x = \pm\sqrt{-25}$

$x = \pm 5i$

II



$P(x) = x^4 + 2x^3 + 22x^2 + 50x - 75 = (x-1)(x+3)(x-5i)(x+5i)$