(3.3 skipped)

$\boxed{3.4}$ Graphs of Polynomials

polynomial : $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0 = 0$

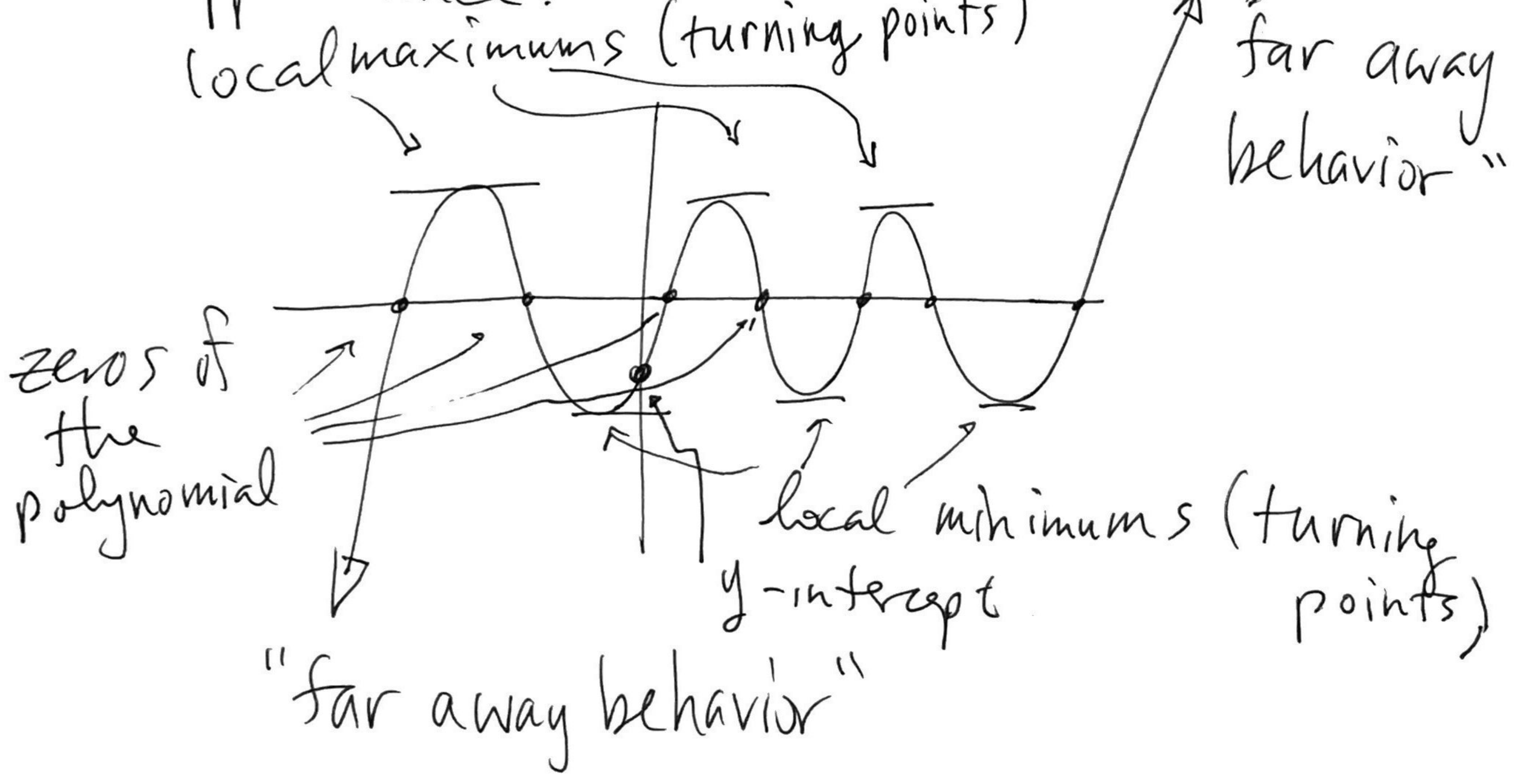
Ex: $4x^6 - x^3 + 11x^2 - 14 = 0$

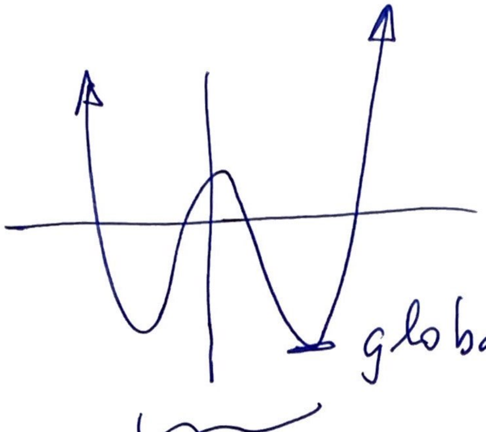
* no fractional or decimal powers.

* "degree" = highest degree of all terms in the polynomial.

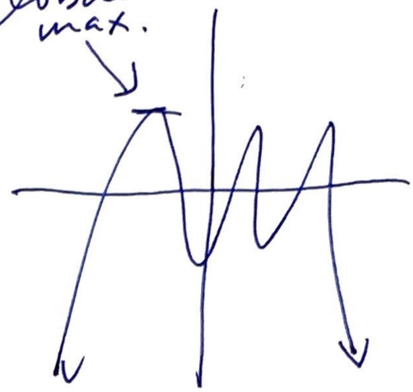
* decreasing power (please)

* generally polynomials have a <u>wooden roller coaster</u> appearance:

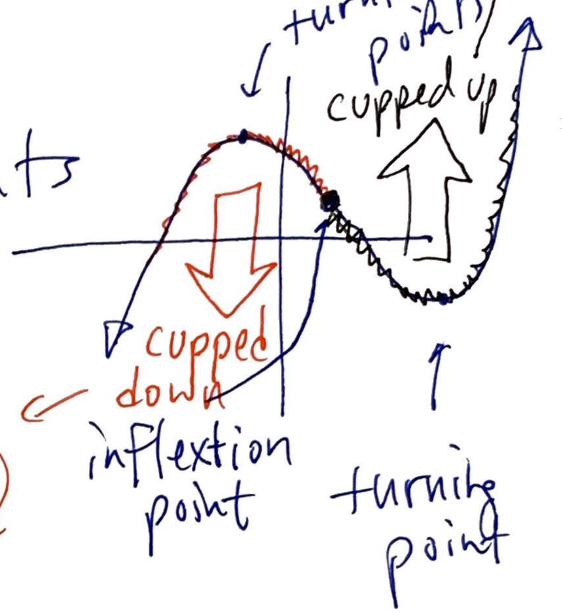




or

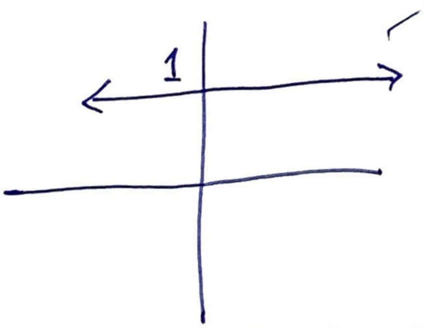


* calculus " inflextion points



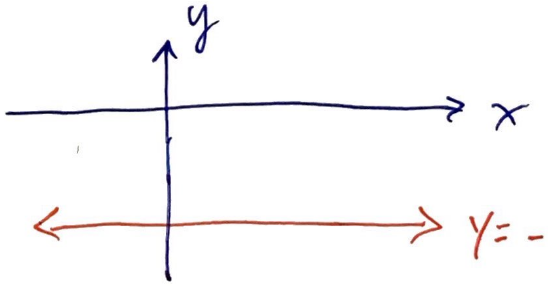
* generalities

$n=0$



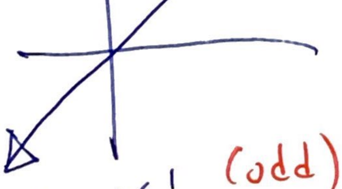
$y = x^0$ (even)

constant



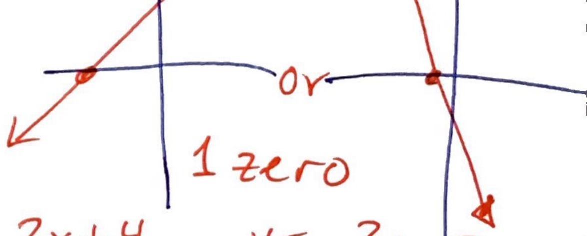
0 zeros

$n=1$



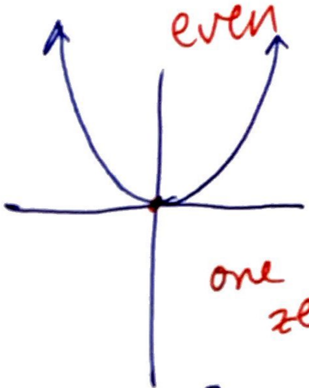
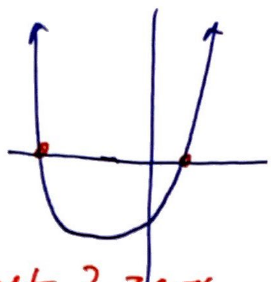
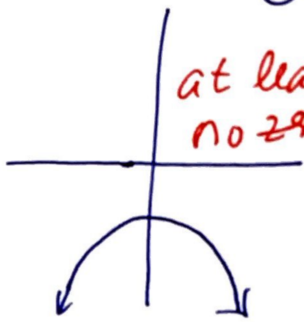
$y = x^1$ (odd)

linear



$y = 2x+4$ $y = -3x-2$

even

$n=2$  quadratic  

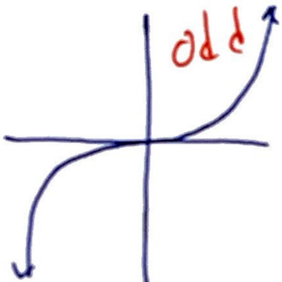
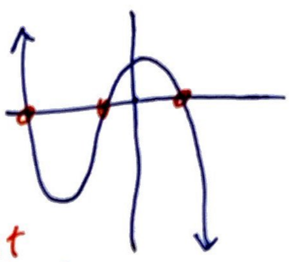
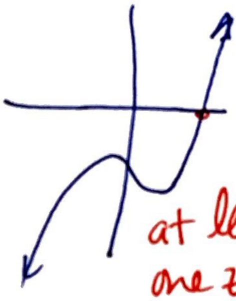
one zero

at most 2 zeros

at least no zeros

$y=x^2$ $\qquad y=2x^2-x-3$ $\qquad y=-x^2-2$

odd

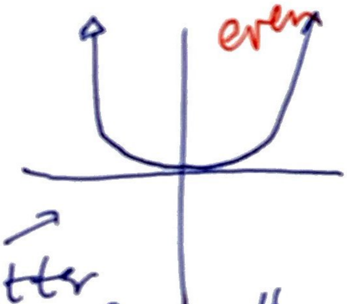
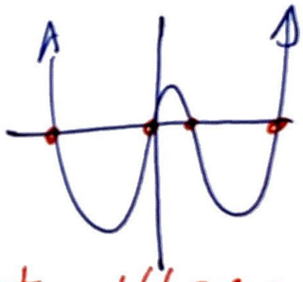
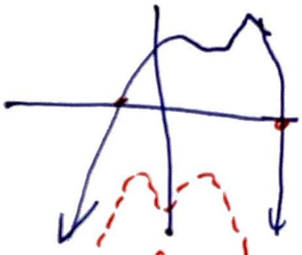
$n=3$  cubic  

at most 3 zeros

at least one zero

$y=x^3$ $\qquad y=-x^3+x+2$ $\qquad y=x^3-4x-3$

even

$n=4$  quartic  

flatter bottoms

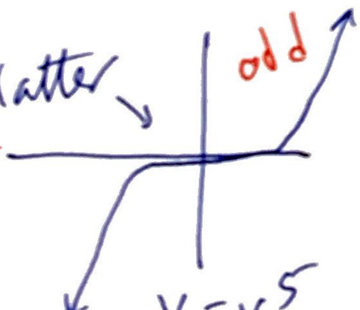
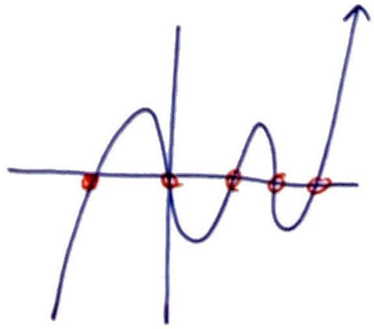
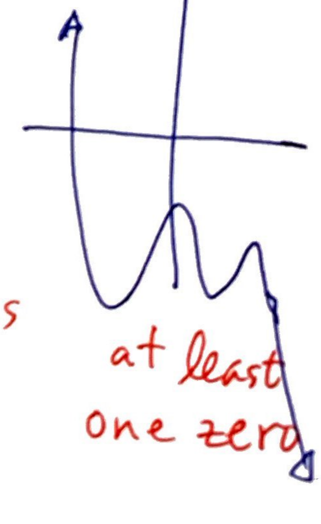
$y=x^4$

at most 4 zeros

at least no zeros

flatter

odd

$n=5$  quintic  

$y=x^5$

at most 5 zeros

at least one zero

⋮

* factored form

Fundamental Thm of Algebra

all polynomials of degree n, that is

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x^1 + a_0 x^0 =$$

can be factored into n = binomials

$$f(x) = a(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_{n-1})(x - r_n)$$

where r_i can be real, zero or complex conjugate or irrational conjugate

$\boxed{\text{EX}}$ $f(x) = x^6 - 2x^4 - 3x^2$

Q: what are the zeros and y-int

* factor

$$f(x) = x^2(x^4 - 2x^2 - 3)$$

$$= x^2(x^2+1)(x^2-3)$$

complex conjugates

irrational conjugates

$$f(x) = x^2 (x+i)(x-i)(x+\sqrt{3})(x-\sqrt{3})$$

↑ multiplicity 2, the rest are multiplicity one

- Real zeros are [x-int] $\begin{cases} x = 0 & (\text{multiplicity } 2) \\ x = \pm\sqrt{3} \end{cases}$

- Complex zeros are $x = \pm i$

- y-int : let $x=0$ $\Rightarrow f(0) = 0^6 - 2\cdot 0^4 - 3\cdot 0^2 = \boxed{0}$

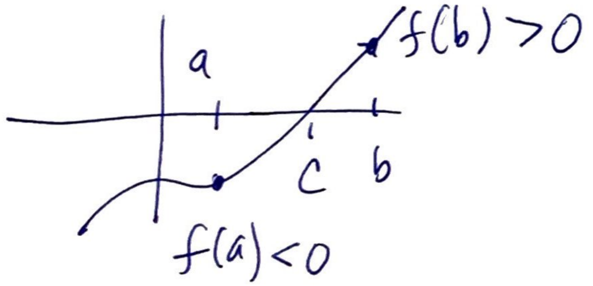
EX what are the zeros of

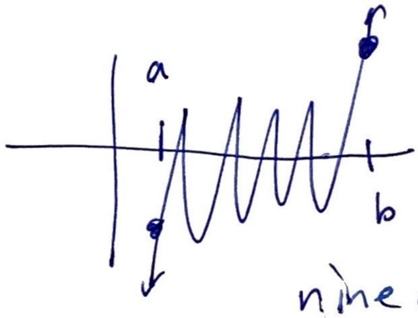
$$f(x) = 2x^3 - x^2 - 8x + 4$$
$$= x^2(2x-1) - 4(2x-1)$$
$$= (2x-1)[x^2-4]$$
$$\boxed{f(x) = (2x-1)(x+2)(x-2)}$$

$2x-1=0$	$x+2=0$	$x-2=0$
$\boxed{x = 1/2}$	$\boxed{x=-2}$	$\boxed{x=2}$

* Intermediate Value Theorem

Let $f(x)$ be a polynomial function. The I.V.T. states that if $f(a)$ and $f(b)$ have opposite signs then there is at least one value "c" between a and b such that $f(c)=0$





nine crossings

EX Confirm there exists a zero between $x=2$ & $x=4$ for $f(x) = x^3 - 9x$

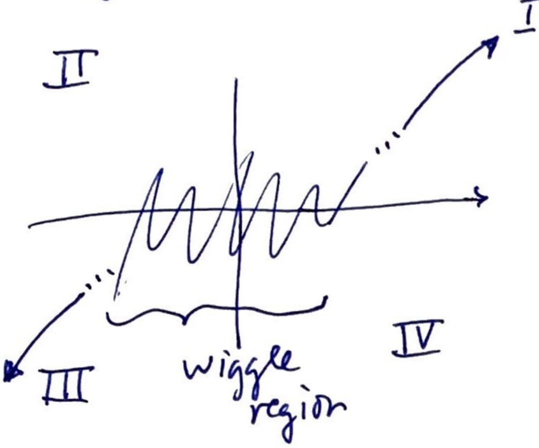
- $f(2) = 2^3 - 9\cdot 2 = 8 - 18 = -10$ (−)
- $f(4) = 4^3 - 9\cdot 4 = 64 - 36 = 28$ (+)

"there exists"

an element of

so, yes, $\exists$ at least one $x \in [2,4]$ where $f(x)=0$.

* Far away behavior : for large x (+) or large x (−)



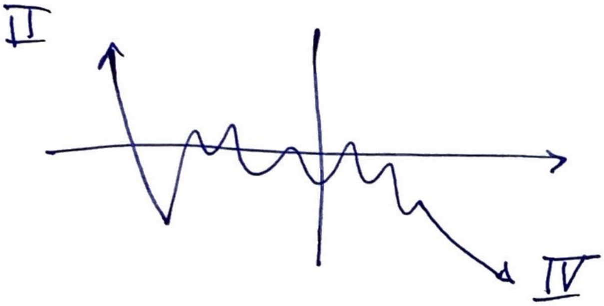
$f(x) = a_n x^n + \cdots + a_0$

y-int. BTW

odd degree $n = 1, 3, 5, \dots$

$a_n > 0$

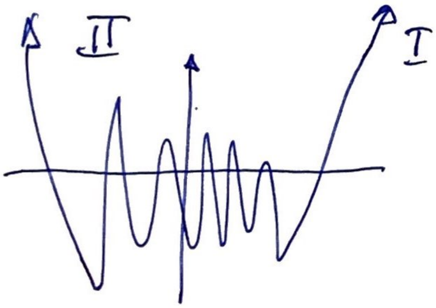
starts in III ends in I



$a_n < 0$

starts in II ends in IV

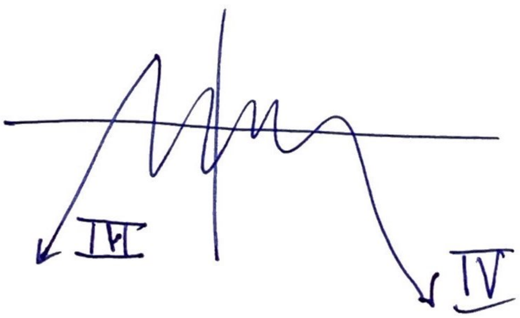
even degree $n = 0, 2, 4, \dots$



if $a_n > 0$

starts in II & leaves in I

if $a_n < 0$

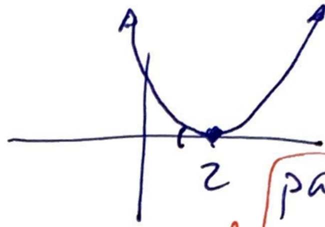


if $a_n < 0$

starts in III & ends IV

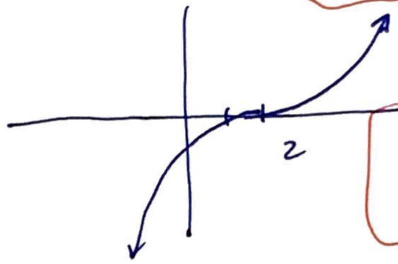
* local crossing behavior

$f(x) = (x-2)^2$



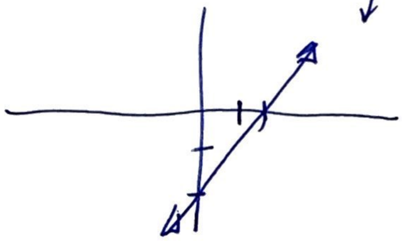
parabolic
near $x=2$
"touch-n-go"

$f(x) = (x-2)^3$



cubic like
near $x=2$

$f(x) = (x-2)$



line-like
near $x=2$

So then

$f(x) = (x-2)(x-1)^2(x)^3$ would show the following

$x=0$ multiplicity 3 cubic-like crossing

$x=1$ multiplicity 2 parabolic-like crossing

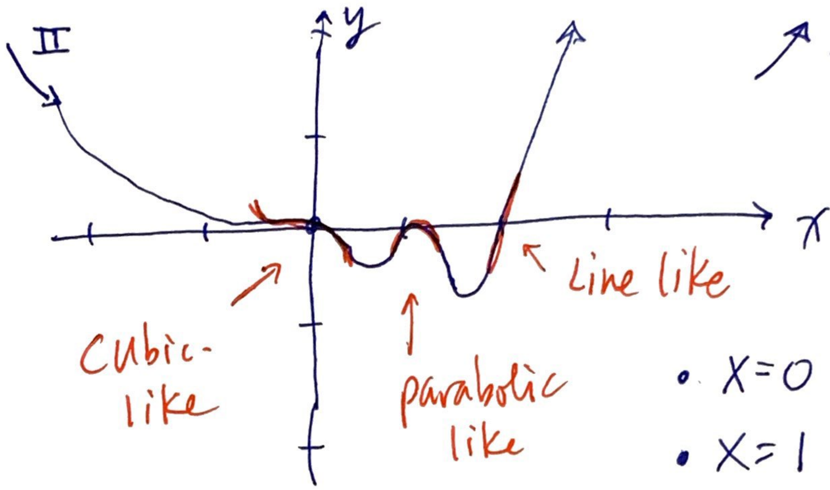
$x=2$ multiplicity 1 line-like crossing

$\boxed{\text{Ex}}$ sketch $f(x) = (x-2)(x-1)^2 x^3 = \underline{x^6 + \cdots + 0}$

• end behavior

II

$\nearrow$ I



• $x=0$ cubic-like
• $x=1$ touch-n-go
• $x=2$ linear-like

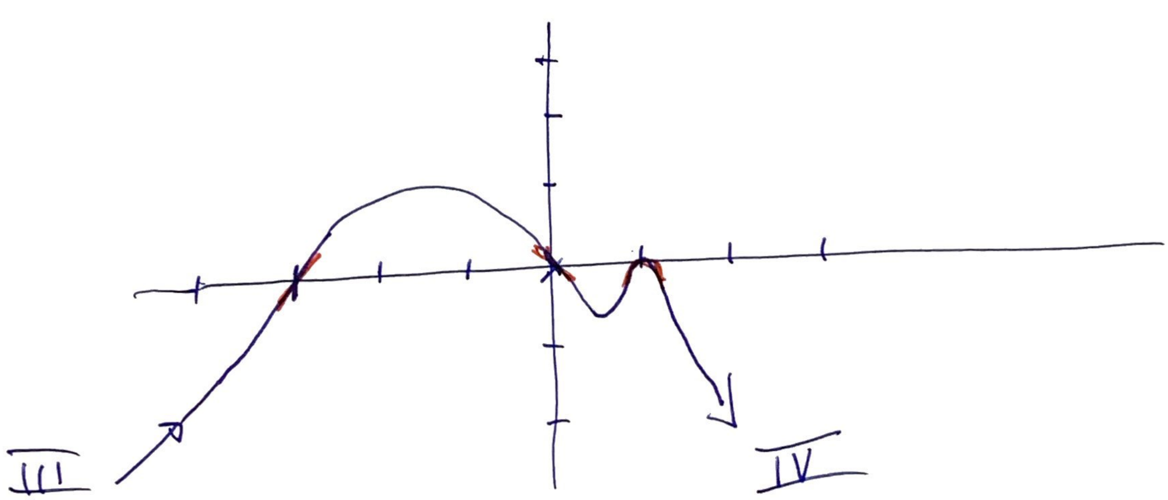
$$x^3(x-2)(x-1)^2 = x^3(x-2)(x^2-2x+1)$$
$$= x^3\left[x^3-2x^2+x-2x^2+4x-2\right]$$
$$\boxed{f(x) = x^6 - 4x^5 + 5x^4 - 2x^3}$$

$\boxed{\text{Ex}}$ sketch $m(x) = -2x(x-1)^2(x+3) = -2x^4 + \cdots x$

• $x=0$ linear, mult is 1
• $x=1$ parabolic, mult is 2
• $x=-3$ linear, mult. is 1

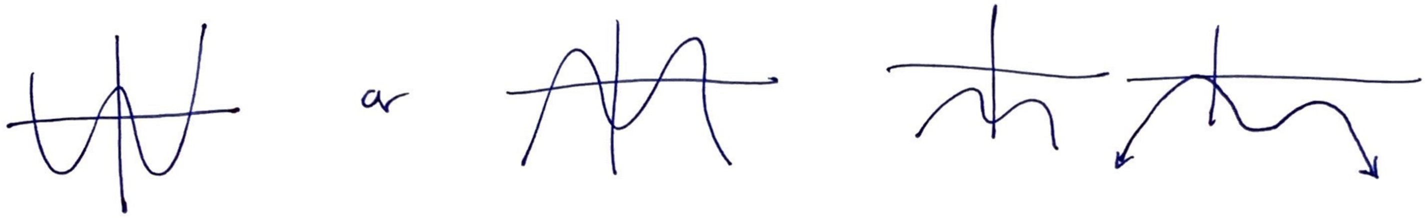
• far away end behavior

III $\to$ IV



* C.S.I problem

$\boxed{\text{Ex}}$ what is the eqn of a degree 4 polynomial that has a multiplicity 2 zero @ $x=4$ and multiplicity 1 zeros @ $x=1$ and $x=-2$. It also has a y-intercept @ 3.



$f(x) = a(x-4)^2(x-1)(x+2)$ Form

$\underbrace{f(0)} = a(0-4)^2(0-1)(0+2)$ Point

$3 = a \cdot 16 \cdot (-1) \cdot 2)$

$3 = -32a \implies a = -\frac{3}{32}$ Solve

So $\boxed{f(x) = -\frac{3}{32}(x-4)^2(x-1)(x+2)}$ Final

3.4 is done

$\boxed{3.5}$ Dividing Polynomials

Review $\frac{37}{3}$.

$$\begin{array}{r} 12 \; r\,1 \\ 3 \overline{)\,37} \\ -3\downarrow \\ \hline 07 \\ -6 \\ \hline 1 \end{array}$$

Ans: $\boxed{\frac{37}{3} = 12 + \frac{1}{3}}$

*Now we do the same for polynomials

$\boxed{\text{Ex}}$ $\frac{2x^2-9x-5}{x-5}$

$$\begin{array}{r} 2x+1 \quad r.\,0 \\ x-5 \overline{)\,2x^2-9x-5} \\ -(2x^2-10x)\downarrow \\ \hline x-5 \\ -(x-5) \\ \hline 0 \end{array}$$

Ans:

$$\boxed{\frac{2x^2-9x-5}{x-5} = 2x+1}$$

BTW: $2x^2-9x-5 = \underline{\underline{(2x+1)(x-5)}}$

we have factored this poly since $r=0$

EX long divide to evaluate the quotient

$$\frac{x^3-126}{x-5}$$

$$
\begin{array}{r}
x^2+5x+25 \quad r. -1 \\
x-5 \,\big)\, \overline{x^3+0x^2+0x-126} \\
-(x^3-5x^2) \qquad\qquad \\
\hline
5x^2+0x \qquad\quad \\
-(5x^2-25x) \qquad \\
\hline
25x-126 \\
-(25x-125) \\
\hline
-1
\end{array}
$$

Ans: $\boxed{\dfrac{x^3-126}{x-5} = x^2+5x+25+\dfrac{-1}{x-5}}$

EX $$\frac{x^4-3x^2+5x-11}{x^2-1}$$

$$
\begin{array}{r}
x^2-2 \quad r.\ 5x-13 \\
x^2+0x-1 \,\big)\, \overline{x^4+0x^3-3x^2+5x-11} \\
-(x^4+0x^3-x^2) \qquad\qquad \\
\hline
-2x^2+5x-11 \\
-(-2x^2+0x+2) \\
\hline
5x-13
\end{array}
$$

Ans: $\boxed{\dfrac{x^4-3x^2+5x-11}{x^2-1} = x^2-2+\dfrac{5x-13}{x^2-1}}$

* when the denominator is a linear binomial, we can use a short cut method call synthetic division

---

EX $\dfrac{3x^3-2x^2+x-4}{x+3}$

$$
\begin{array}{r}
3x^2-11x+34 \\
x+3\,\overline{)\,3x^3-2x^2+x-4} \\
-(3x^3+9x^2) \qquad\quad \\
\hline
-11x^2+x \qquad \\
-(-11x^2-33x) \qquad \\
\hline
34x-4 \\
-(34x+102) \\
\hline
-106
\end{array}
$$

$$\boxed{\frac{3x^3-2x^2+x-4}{x+3} = 3x^2-11x+34-\frac{106}{x+3}}$$

or $\boxed{3x^3-2x^2+x-4 = (3x^2-11x+34)(x+3)-106}$

* synthetic div. (denominator must be linear)

$$
\begin{array}{c|cccc}
 & x^3 & x^2 & x^1 & x^0 \\
-3 & 3 & -2 & 1 & -4 \\
 & & -9 & 33 & -102 \\
\hline
 & 3 & -11 & 34 & \boxed{-106} \\
 & x^2 & x^1 & x^0 & r
\end{array}
$$

change the sign (x+3)

← reduced poly

remainder

$$\boxed{\frac{3x^3-2x^2+x-4}{x+3} = 3x^2-11x+34-\frac{106}{x+3}}$$

$$\underbrace{3x^3 - 2x^2 + x} - 4$$

$$= x\,(\underbrace{3x^2 - 2x} + 1) - 4$$

$$= x\,(x\,(3x-2) + 1) - 4$$

$x+3=0$

$x=-3$

$$f(-3) = -3\,(-3\,(3\cdot(-3) - 2)) + 1) - 4$$

$$-3\,(\underbrace{-9 - 2})$$

$$\underbrace{-3\,(\;-11\;) + 1}$$

$$-3\,(\quad 34 \quad) - 4$$

not finished with 3.5 (but almost)