$\boxed{11.4}$ Summation

In 11.1 to 11.3 we investigated sequences.
Now we start to add the terms of a sequence together to get a "series."

$\boxed{\text{Ex}}$ For $\left\{\frac{1}{n+1}\right\}$, or $a_n = \frac{1}{n+1}$, lets write out the sum:

$$S_n = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \cdots + \frac{1}{n+1}$$

S_n is refered to as the <u>partial sum</u>.

$\boxed{\text{Ex}}$ in the sequence above write S_5 out:

$$S_5 = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} + \frac{1}{5+1}$$

$$S_5 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

$$= \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} + \frac{10}{60}$$

$$\boxed{S_5 = \frac{87}{60}}$$

summation notation Σ the greek capital sigma letter for S (Σ, σ)

Notation $\boxed{S_5 = \sum_{n=1}^{5}\left(\frac{1}{n+1}\right)}$

Nomenclature: n = index 5 = ending index
1 = starting index

* The arithmetic series

We start our study of summation with the arithmetic sum.

The sequence $\{a_1 + (n-1)d\}$ can be summed up as:

$$S_N = a_1 + (a_1+d) + (a_1+2d) + (a_1+3d) + \dots + a_1 + (N-1)d$$

- We can actually derive a formula for this sum by rewritting the sum starting from the last term and decrementing each term by "d":

$$S_N = a_N + (a_N - d) + (a_N - 2d) + (a_N - 3d) + \dots + a_N - (N-1)d$$

- Next we ADD these two sums together:

$$S_N + S_N = (a_1 + a_N) + (a_1 + d + a_N - d) + (a_1 + 2d + a_N - 2d) + \dots + (a_1 + (N-1)d + a_N - (N-1)d)$$

- we see all terms with "d" cancell!

$$2S_N = (a_1 + a_N) + (a_1 + a_N) + \dots + (a_1 + a_N)$$

$$2S_N = N \cdot (a_1 + a_N)$$

$$\Rightarrow \boxed{S_N = N \cdot \left(\frac{a_1 + a_N}{2}\right)}$$

average the 1st & last terms then multiply by the number of terms present.

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$\boxed{9}$ Use the formula $S_N = \dfrac{N(a_1+a_2)}{2}$

to sum the numbers 1 to 100

i.e. $1+2+3+4+\cdots+98+99+100$

$$S_N = \frac{N(a_1+a_N)}{2} \quad \begin{cases} a_1 = 1 \\ a_{100} = 100 \end{cases}, \; N = 100$$

$$S_{100} = \frac{100(1+100)}{2}$$

$$S_{100} = 50(101)$$

$$S_{100} = 5000+50 = \boxed{5050}$$

⊛ Euler in Kindergarten:

$$S_{100} = 1+2+3+4+\cdots+48+49+50+51+52+\cdots+97+98+99+\boxed{100}$$

100 100 100 100 100 100

$$= 49 \cdot 100 + 50 + 100$$

$$\boxed{S_{100}} = 4900+100+50 = \boxed{5050}$$

③

$\boxed{\text{EX}}$ Sum the first 5 terms of the arithmetic sequence $\{3, 9, 15, 21, 27, \dots\}$

$n = 1, 2, 3, 4, 5$

(i) data: $N = 5$, $a_1 = 3$, $a_5 = 27$

(ii) via formula

$$S_5 = \frac{5 \cdot (3+27)}{2}$$

$$\boxed{S_5 = 75}$$

30

30 ← Euler's trick (later on)

BTW: by brute force

$$S_5 = 3 + 9 + 15 + 21 + 27$$

$$\boxed{S_5 = 75}$$

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* We can recast the arithmetic sum formula to use the 1^{st} term and common difference:

$$S_N = \frac{N(a_1 + a_N)}{2}, \text{ but } a_N = a_1 + (N-1)d$$

$$= \frac{N(a_1 + a_1 + (N-1)d)}{2}$$

$$S_N = \sum_{n=1}^{n=N} a_n$$

$$= \frac{N}{2}(2a_1 + (N-1)d)$$

$$\boxed{S_N = \frac{N(N-1)d}{2} + Na_1}$$

$$\boxed{S_N = N\left(a_1 + \frac{N-1}{2}d\right)}$$

use any version that works for you

← I like this one ... sorta like the original formula

$\boxed{EX}$ Consider the arithmetic sequence $\{\frac{1}{2}, \frac{4}{2}, \frac{7}{2}, \frac{10}{2}, \dots\}$. Find the sum of the first 19 terms.

$a_N = a_1 + (N-1)d$

(i) data: $d = 3/2$, $a_1 = \frac{1}{2}$, $N = 19$

(ii) formula $S_{19} = \frac{19(19-1)(\frac{3}{2})}{2} + 19(\frac{1}{2})$

$$= \frac{19 \cdot 9}{2} + \frac{19}{2} = \frac{19}{2}(9+1) = \frac{19}{2} \cdot 10$$

$$\boxed{S_{19} = 95}$$

Geometric Series Now lets sum up geom sequences:

- Recall $\{a_1 (r)^{n-1}\}$ or $\underline{a_n = a_1 r^{n-1}}$
- The trick here is to multiply the partial sum by "r" and subtract:

$$S_N = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{\cdots} + \cancel{a_1 r^{N-1}}$$

these all cancell out

$$\ominus\; r S_N = \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cancel{\cdots} + a_1 r^N$$

$$S_N - r S_N = a_1 - a_1 r^N$$

$$S_N (1 - r) = a_1 (1 - r^N)$$

$$\boxed{S_N = a_1 \left(\frac{1 - r^N}{1 - r} \right)}$$

Sum of geometric series.
a_1 = 1st term
r = common ratio

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Sum using the formula

$\boxed{10}$ $9+3+1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$

(i) data: $a_1=9$, $r=\frac{1}{3}$, $N=7$

(ii) $S_7 = 9\left(\frac{1-(\frac{1}{3})^7}{1-\frac{1}{3}}\right)$

$= \frac{9\ (3^7-1)}{\frac{2}{3}\ 3^7} = \frac{27}{2}\left(\frac{2187-1}{2187}\right)$

$=$

$\boxed{11}$ Sum of the series $2+1.6+1.28+1.024+\cdots$

$2 \xrightarrow{*0.8} 1.6 \xrightarrow{*0.8} 1.28 \xrightarrow{*0.8} 1.024$

(i) data: $a_1=2$, $r=0.8$, ∞

(ii) $S = a_1\left(\frac{1}{1-r}\right)$

$= 2\left(\frac{1}{1-0.8}\right) = 2\left(\frac{1}{0.2}\right) = 2\left(\frac{1}{\frac{2}{10}}\right)$

$= 2\left(\frac{10}{2}\right) = \boxed{10}$

EX Find the sum of the first 5 terms of $\left\{\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \frac{1}{256}, \ldots\right\}$

(i) data: $a_1 = 1/3$, $r = \frac{1}{4}$, $N = 5$

(ii) formula: $S_5 = \frac{1}{3}\left(\frac{1-(1/4)^5}{1-(1/4)}\right)$

$$S_5 = \frac{1}{3}\left(\frac{1-\frac{1}{1024}}{3/4}\right)$$

$$= \frac{1}{3}\left(\frac{1023}{3/4 \cdot 1024}\right) = \frac{\overset{\cancel{2}}{\cancel{4}}}{\underset{3}{\cancel{9}}} \cdot \frac{\overset{341}{\cancel{1023}}}{\underset{\substack{\cancel{512}\\256}}{\cancel{1024}}}$$

$$\boxed{S_5 = \frac{341}{768}}$$

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r^N vanishes to yeild the formula

$$\boxed{S_\infty = a_1\left(\frac{1}{1-r}\right)}$$

- but r^N vanishes if r is < 1, and to be technical we need to include r between -1 and 1

i.e. we need r to satisfy

$\boxed{-1 < r < 1}$ We abbreviate this as $\boxed{|r| < 1}$

$\boxed{\text{EX}}$ Find the sum of the first 5 terms of $\left\{\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \frac{1}{256}, \ldots\right\}$

(i) data: $a_1 = 1/3$, $r = \frac{1}{4}$, $N = 5$

(ii) formula: $S_5 = \frac{1}{3}\left(\frac{1-(1/4)^5}{1-(1/4)}\right)$

$$S_5 = \frac{1}{3}\left(\frac{1-\frac{1}{1024}}{3/4}\right)$$

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$$\boxed{S_5 = \frac{341}{768}}$$

Infinite series

Play around
- Observation #1: $(((1.3)^2)^2)^2 \cdots = \text{big}\#.$
- Observation #2: $(((0.8)^2)^2)^2 \cdots = 0$

This leads use to conclude that, as N gets very large

$$\boxed{\begin{array}{l} r^N \xrightarrow{\text{approaches}} \infty \quad \text{if } r > 1 \\ \text{and } r^N \longrightarrow 0 \quad \text{if } r < 1 \end{array}}$$

Should we choose to let a geometric series go on forever we can alter our formula

$$S_N = a_1 \left(\frac{1 - r^N}{1 - r} \right)$$

for larger and larger N and notice that r^N vanishes to yeild the formula

$$\boxed{S_\infty = a_1 \left(\frac{1}{1-r} \right)}$$

- but r^N vanishes if r is < 1, and to be technical we need to include r between -1 and 1

i.e. we need r to satisfy

$$-1 < r < 1$$

We abbreviate this as

$$|r| < 1$$

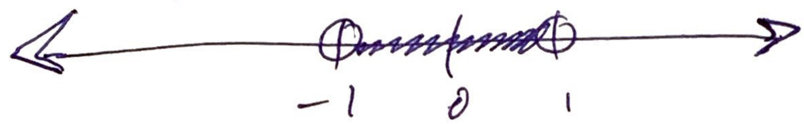
SUMMARY:

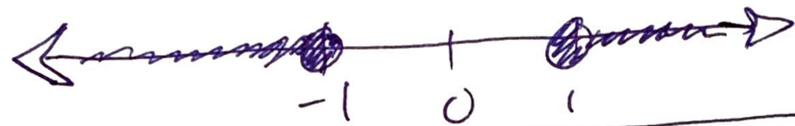
$$\boxed{S_\infty = a_1\left(\frac{1}{1-r}\right) \text{ if } |r|<1}$$

- the sum "converges" if $\underline{-1<r<1}$
- the sum "diverges" if $\underline{|r|\geq 1}$

ie $\{r \mid (-\infty,-1) \cup (1,\infty)\}$

- Graphical representation of "r":

r:  Convergence

r:  divergeges to ∞

$\boxed{\text{EX}}$ Find the sum of $\sum_{n=1}^{\infty} -\left(-\frac{1}{2}\right)^{n-1}$ if it exists.

$= -1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} \ldots$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{alternating series}}$

(i) data: $a_1 = -1$, $r = -\frac{1}{2}$, ∞

(ii) formula: $S = a_1\left(\frac{1}{1-r}\right)$ if $|r|<1$

which it is for $r = -1/2$

So

$$S = -1\left(\frac{1}{1-(-\frac{1}{2})}\right)$$

$$\boxed{S = -\frac{2}{3}}$$

Application: Repeating decimals

- If an infinitely long decimal number has no repetition it belongs to the irrational numbers.
- If an infinitely long decimal number repeats in a pattern for ever it is a rational number.
- Rational numbers can be written as a ratio.

Q: How do we express 7.11111 ⋯ as a fraction?

A: Rewrite as 7.111111 ⋯ as

$$7 + 0.1 + 0.01 + 0.001 + 0.0001 + \cdots$$

$$= 7 + \underbrace{\frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \left(\frac{1}{10}\right)^4 + \cdots}_{\text{geometric series!}}$$

$$a_1 = \frac{1}{10}, \quad r = \frac{1}{10}$$

$$S_\infty = \frac{1}{10}\left(\frac{1}{1 - 1/10}\right)$$

$$= \frac{1}{10} \cdot \frac{1}{9/10}$$

$$= \frac{1}{10} \cdot \frac{10}{9} = \frac{1}{9}$$

So 7.11111 ⋯

$$= 7 + \frac{1}{9}$$

$$= \boxed{64/9}$$ a rational number after all!

$\boxed{\text{Ex}}$ Write 11.123 123 123 ⋯ as a fraction

$$11 + 0.123 + 0.000123 + 0.000\,000\,123 + 0.000\,000\,000\,123 + \cdots$$

$$= 11 + 123\left(\frac{1}{1000}\right) + 123\left(\frac{1}{1000}\right)^2 + 123\left(\frac{1}{1000}\right)^3 + \cdots$$

$$= 11 + 123\left\{\frac{1}{1000} + \left(\frac{1}{1000}\right)^2 + \left(\frac{1}{1000}\right)^3 + \cdots\right\}$$

$\hookrightarrow a_1 = \frac{1}{1000}, \quad r = \frac{1}{1000}$

$$S_\infty = \frac{1}{1000}\left(\frac{1}{1 - \frac{1}{1000}}\right) = \frac{1}{1000}\left(\frac{1}{\frac{999}{1000}}\right) = \frac{1}{999}$$

$$= 11 + 123\left\{\frac{1}{999}\right\}$$

$$= 11 + \frac{123}{999} = \frac{11 \cdot 999 + 123}{999} = \boxed{\frac{11112}{999}}$$

⊛ In general if we have 0.abcdabcdabcd ⋯ then this is equal to $\frac{abcd}{9999}$ ← divide by the length of the repetition but all "9"s.

$\boxed{\text{Ex}}$

6.787878 ⋯	7.111 ⋯	14.5134513451 34 ⋯	etc.
$= 6 + \frac{78}{99}$	$= 7 + \frac{1}{9}$	$= 14 + \frac{5134}{9999}$	