$\boxed{\text{Chapte 10}}$: Conic Sections (Analytic Geom) ①

$\boxed{10.1}$ Ellipses

- parametric equations $\begin{cases} x(\theta) = r\cos\theta \\ y(\theta) = r\sin\theta \end{cases}$

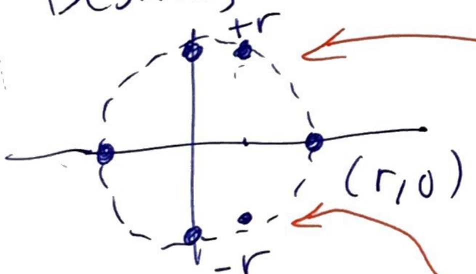
- plug these into
  $\cos^2\theta + \sin^2\theta = 1$ ← identity in trig

$$\boxed{\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1} \Rightarrow \text{circle of radius } r$$

$$\{ x^2 + y^2 = r^2 \}$$

$$y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$$

Desmos



x	y
0	$\pm\sqrt{r}$
r	0
-r	0
r/2	$\pm\sqrt{r^2 - \left(\frac{r}{2}\right)^2} = \pm r\sqrt{1 - \frac{1}{4}} = \pm r\frac{\sqrt{3}}{2}$

we see this is a circle! $\boxed{x^2 + y^2 = r^2}$

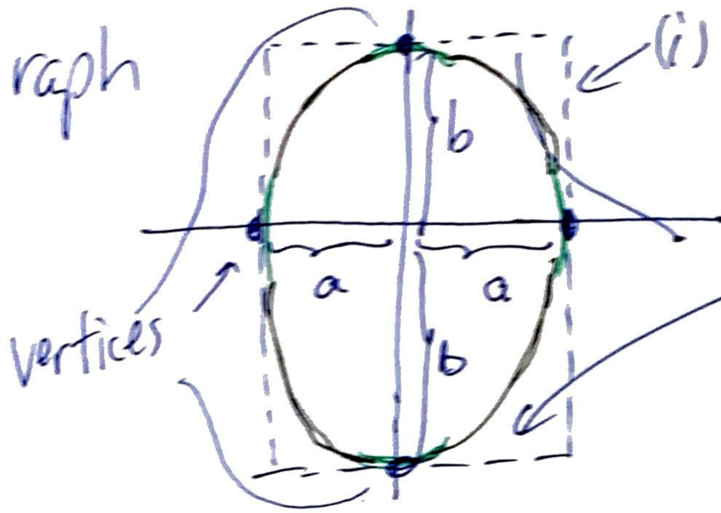
- Consider $\left.\begin{array}{l} x = a\cos\theta \\ y = b\sin\theta \end{array}\right\}$ Desmose $\begin{cases} a = 2 \\ b = 5 \end{cases}$

$$c^2 + s^2 = 1 \Rightarrow \boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1}$$ equation of an ellipse

* Standard form of an ellipse is

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

* graph

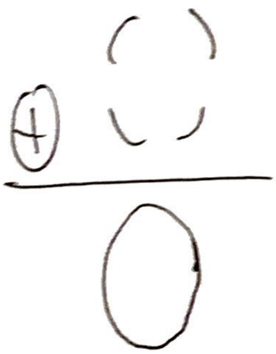


(i) draw walls @ $x=\pm a$

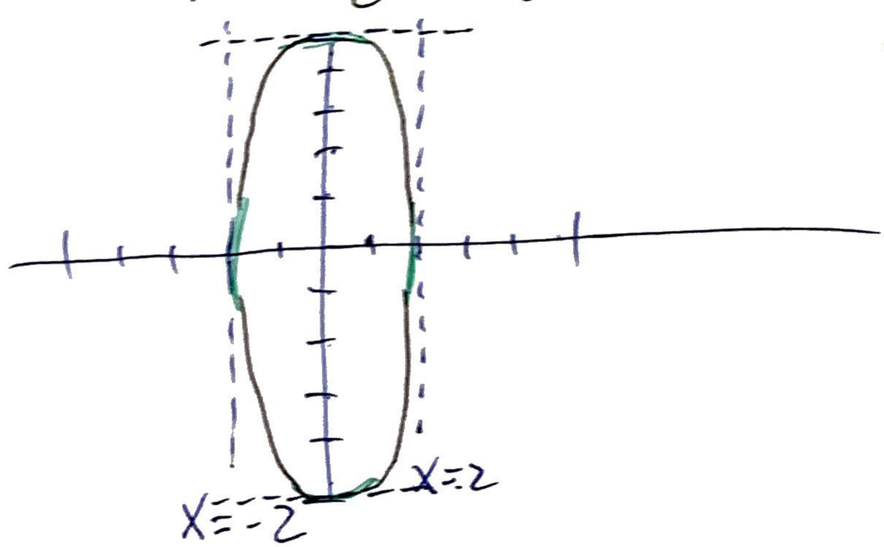
(ii) draw ceiling & floor @ $y=\pm b$

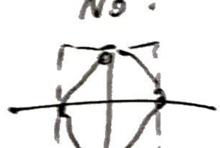
(iii) the ellipse will be tangent to the vertices, so draw tangent segment

(iv) Connect the tangent segments.



EX Graph $\frac{x^2}{2^2}+\frac{y^2}{5^2}=1$





* putting general form ellipses into std form :

Ex $\boxed{9x^2+4y^2-36=0}$ general form

- $\div 36$ $\dfrac{9x^2}{36}+\dfrac{4y^2}{36}-1=0$

- move the 1 over

$$\frac{9x^2}{36}+\frac{4y^2}{36}=1$$

- place the numerators under the denominators

$$\frac{x^2}{\frac{36}{9}}+\frac{y^2}{\frac{36}{4}}=1$$

b/c

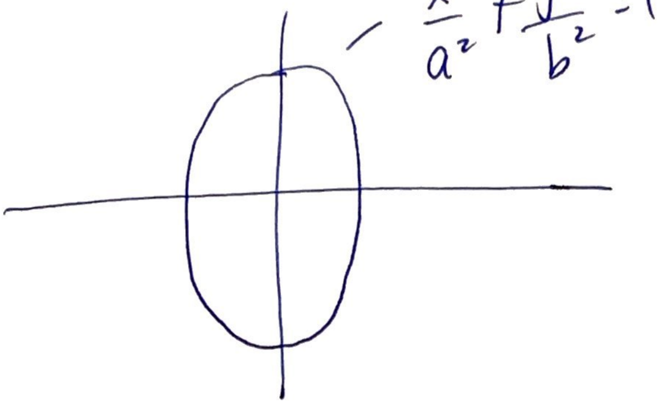
$$\frac{1}{\frac{36}{9}}=\frac{9}{36}$$

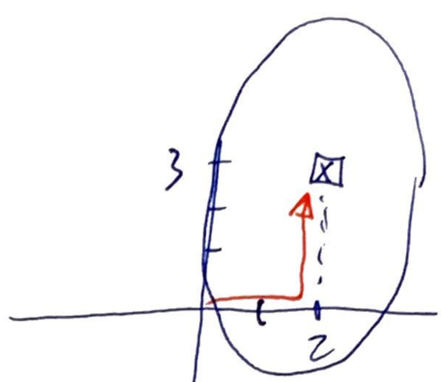
- simplify $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

- convert the denominators to numbers squared.

$$\boxed{\frac{x^2}{2^2}+\frac{y^2}{3^2}=1}$$ std. form

* off-center ellipses





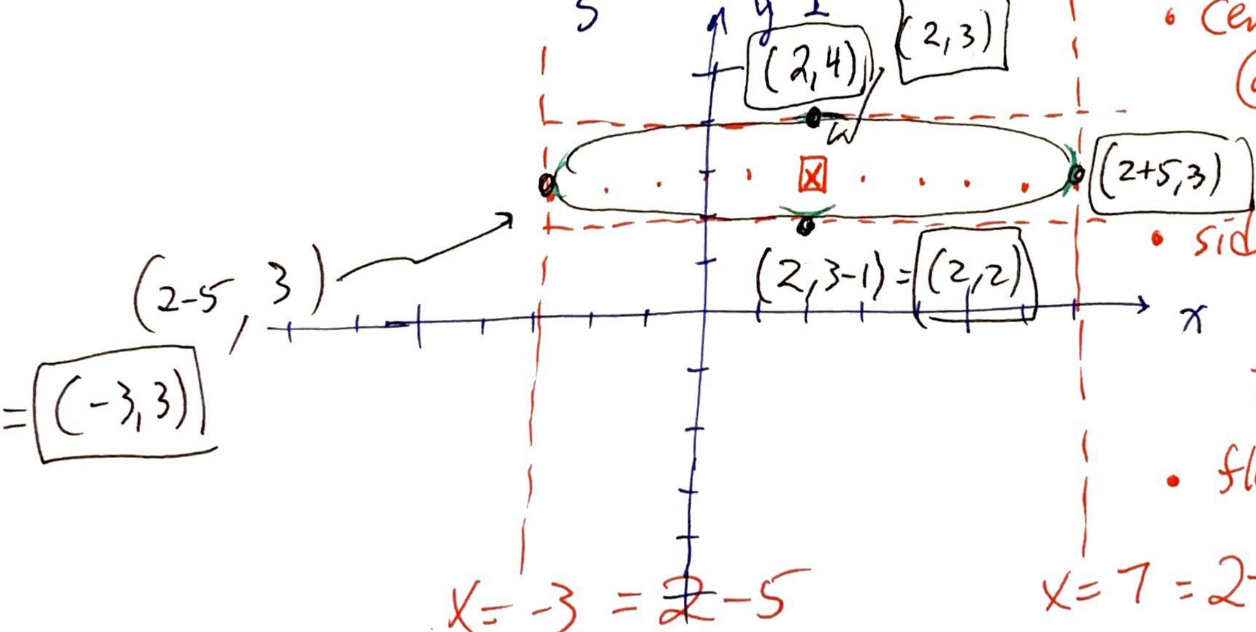
$$\boxed{\frac{(x-2)^2}{a^2}+\frac{(y-3)^2}{b^2}=1}$$

- in general $\boxed{\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1}$ std. form

  - center @ (h,k)

    from this center we have $\pm a$ side walls and $\pm b$ floor & ceiling.

$\boxed{\text{EX}}$ Sketch $\frac{(x-2)^2}{5^2}+\frac{(y-3)^2}{1^2}=1$



- center @ $(2,3)$
- side walls @ ± 5 from the center
- floor/ceiling @ ± 1 from center

* Converting general form to std. form but for ellipses off origin.

$\boxed{\text{EX}}$ $9x^2+72x+16y^2+16y+4=0$

we need to complete the square twice

- factor out the coeff. on x^2 & on y^2

$$9(x^2+8x)+16(y^2+y)+4=0$$

- insert magic zeros

$$9\Big(\underbrace{x^2+8x+\left(\tfrac{8}{2}\right)^2}_{(x+\frac{8}{2})^2}-\left(\tfrac{8}{2}\right)^2\Big)+16\Big(\underbrace{y^2+y+\left(\tfrac{1}{2}\right)^2}_{(y+\frac{1}{2})^2}-\left(\tfrac{1}{2}\right)^2\Big)=-4$$

- complete the square

$$9\left(\left(x+\tfrac{8}{2}\right)^2-\left(\tfrac{8}{2}\right)^2\right)+16\left(\left(y+\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2\right)=-4$$

- simplify

$$9\left((x+4)^2-16\right)+16\left(\left(y+\tfrac{1}{2}\right)^2-\tfrac{1}{4}\right)=-4$$

$$9(x+4)^2-144+16\left(y+\tfrac{1}{2}\right)^2-\cancel{4}=-\cancel{4}$$

$$9(x+4)^2+16\left(y+\tfrac{1}{2}\right)^2=144$$

- divide by 144

$$\frac{9(x+4)^2}{144} + \frac{16\left(y+\frac{1}{2}\right)^2}{144} = 1$$

• flip

$$\frac{(x+4)^2}{\frac{144}{9}} + \frac{(y+1/2)^2}{\frac{144}{16}} = 1$$

• simplify

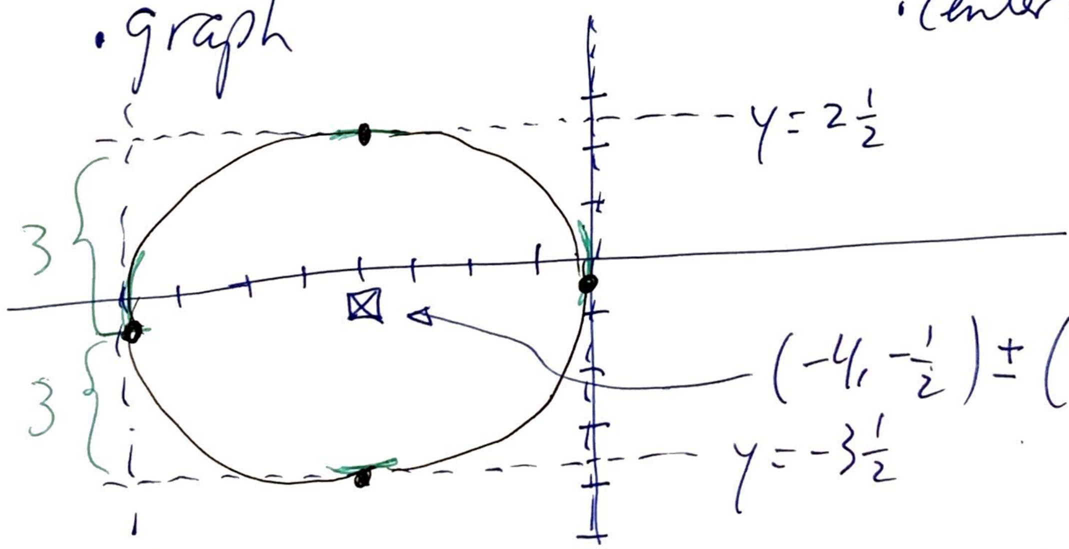
$$\frac{(x+4)^2}{16} + \frac{(y+1/2)^2}{9} = 1$$

$$\boxed{\frac{(x+4)^2}{4^2} + \frac{\left(y+\frac{1}{2}\right)^2}{3^2} = 1}$$

$$9x^2 + 72x + 16y^2 + 16y + 4 = 0$$

• graph

• center @ $\left(-4, -\frac{1}{2}\right)$



$y = 2\frac{1}{2}$

Top & bot

$\left(-4, -\frac{1}{2}\right) \pm (0, 3)$

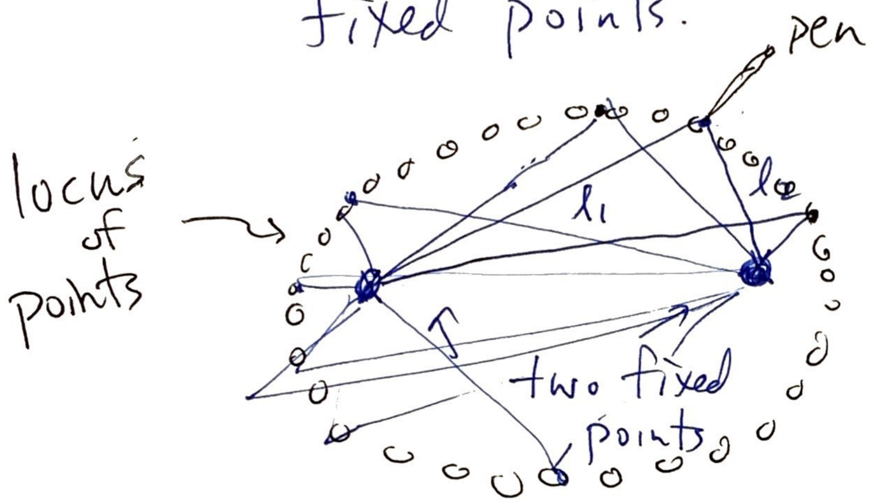
$y = -3\frac{1}{2}$

$x = -8$ $\qquad$ $x = 0$ $\leftarrow \left(-4, -\frac{1}{2}\right) \pm (4, 0)$

4 $\qquad$ 4

* fociii

Analytically, the description of an ellipse is given by the "locus of points" that satisfied the sum of the distance from the point on the ellipse and two fixed points.



$l_1 + l_2$ = fixed numer

These two fixed points are called foci, each being called a focus.

For ellips the location of the foci is $a^2 - b^2 = c^2$, where c is the distance from the center. laying down ellipse

$a > b$

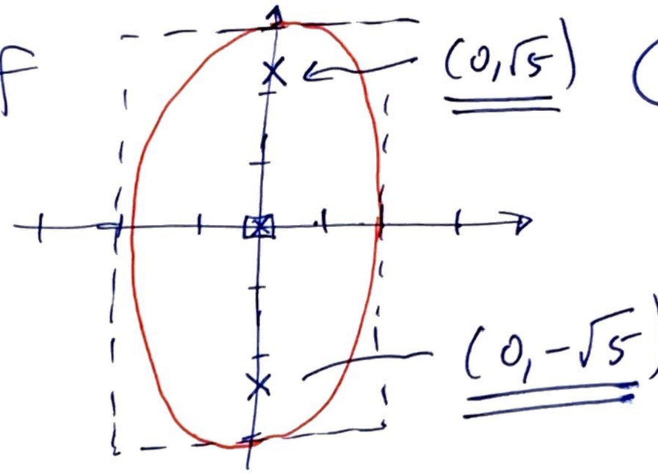
$b^2 - a^2 = c^2$ Standing up ellipse.

$b > a$

$$\boxed{\max^2(a,b) - \min^2(a,b) = c^2}$$

EX Find the foci of

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$



$$\max^2(2,3) - \min^2(2,3) = c^2$$

$$3^2 - 2^2 = c^2$$

$$9 - 4 = c^2$$

$$5 = c^2$$

$$\boxed{c = \pm\sqrt{5}}$$

The foci are alway on the major axis (the longest one).

locations in our example is

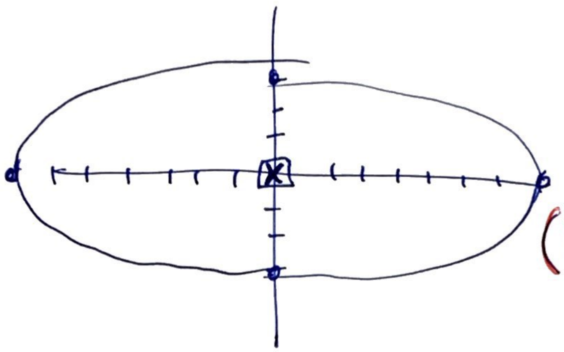
$$\boxed{(0, +c) \text{ and } (0, -c)}$$

$\sqrt{5} \approx 2.2$

* C.S.I. example (indirect approach)

I obtain eqn from the graph

EX



eqn = ?

(i) $$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

• Start with this and then populate this...

(ii) data gathering

$a = 7$
$b = 3$
$h = 0$
$k = 0$

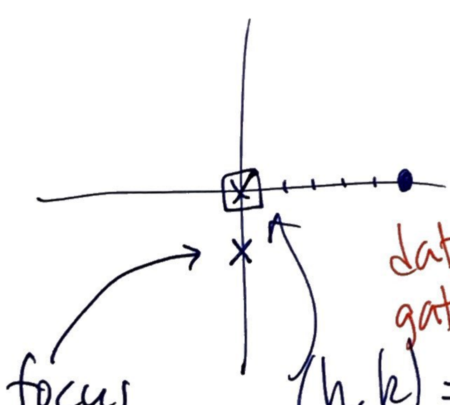
(iii) populate

$$\boxed{\frac{x^2}{7^2} + \frac{y^2}{3^2} = 1}$$

$$\frac{x^2}{5^2} + \frac{y^2}{\sqrt{29}^2} = 1$$

II Find eqn of ellipse if the center is at the origin, focus is @ (0,-2) and a point on the graph is (5,0)

(i) sketch



So (i) eqn

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

data gathering

• since $(h,k) = (0,0)$

• major axis the y axis since the foci are on the y-axis

• extra point is on the minor axis.

$$\frac{x^2}{5^2} + \frac{y^2}{b^2} = 1$$

• foci eqn: $b^2 - a^2 = c^2$ since it is a stand-up ellipse.

$b = \sqrt{c^2 + a^2} = \sqrt{2^2 + 5^2}$

$\boxed{b = \sqrt{29}}$ Done

c/w mod #4 cont.

$|a^2 - b^2| = c^2$

ellipses

$\boxed{\#9}$ Sketch $\dfrac{(x-2)^2}{81} + \dfrac{(y+1)^2}{16} = 1$

label center, all vertices, and foci.

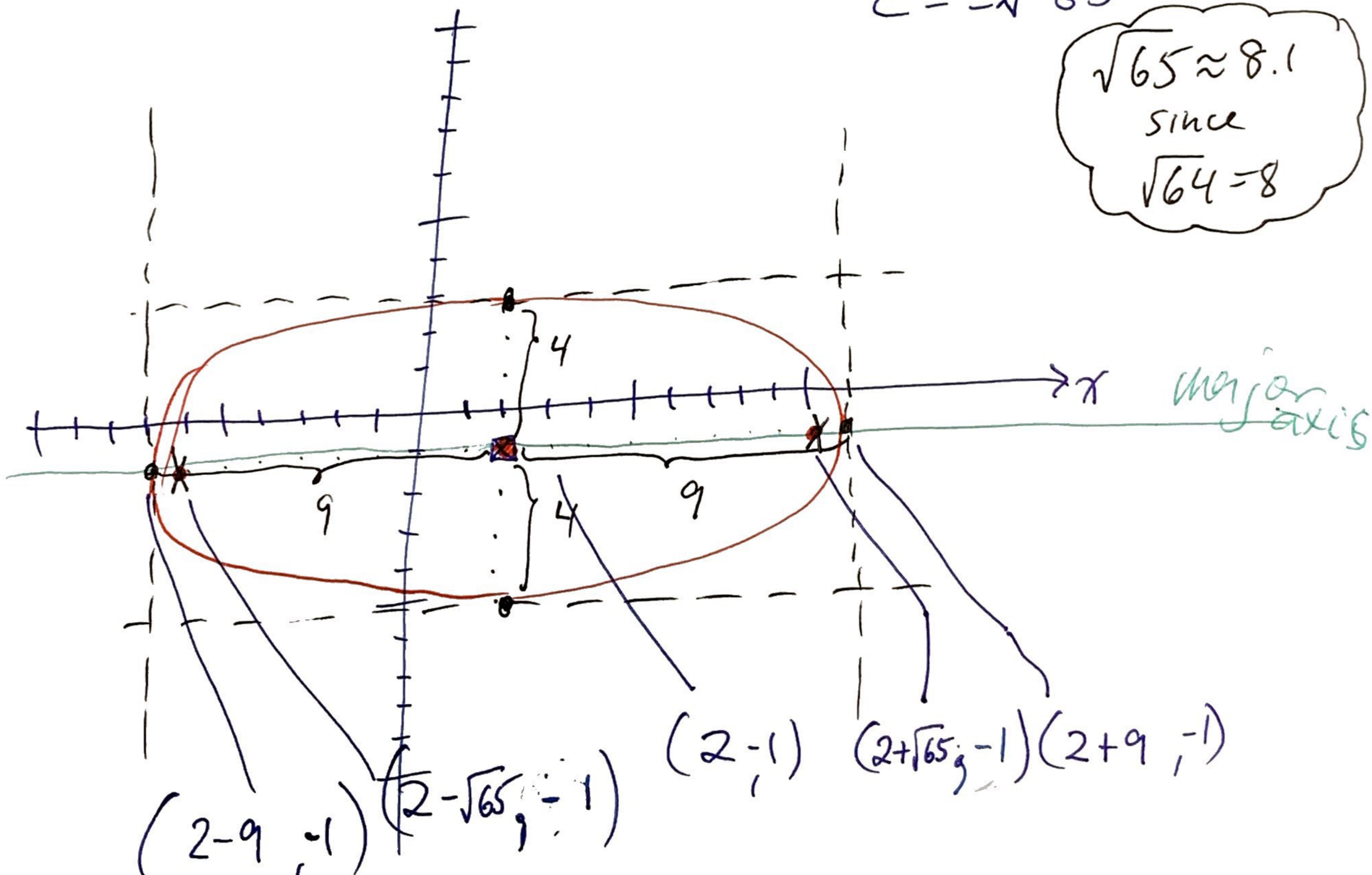
$$\frac{(x-2)^2}{9^2} + \frac{(y+1)^2}{4^2} = 1$$

- $a = 9$, $b = 4$
- center @ $(2, -1)$
- foci @ $\pm\sqrt{9^2 - 4^2}$

$= \pm\sqrt{81 - 16}$

$c = \pm\sqrt{65}$

$\sqrt{65} \approx 8.1$ since $\sqrt{64} = 8$



$(-7, -1)$, $(2-\sqrt{65}, -1)$, $(2, -1)$, $(2+\sqrt{65}, -1)$, $(11, -1)$