Induction

In mathematics, we are not happy with the phrase "it looks like it is..." which is an observation based on a few occurrences of an phenomena.

EX Consider the recursive relationship

$$a_{n+1} = \sqrt{2+a_n}\ , \quad a_1 = \sqrt{2} \leftarrow \text{seed}$$

play around:
$$a_{0+1} = a_1 = \sqrt{2} \approx 1.1414$$

$n=0$: $a_{1+1} = \sqrt{2+a_1} = \sqrt{2+\sqrt{2}} \approx 1.8477$

$n=1$: $\rightarrow$ $a_{2+1} = \sqrt{2+a_2} = \sqrt{2+\sqrt{2+\sqrt{2}}} \approx 1.96$

$n=2$: $a_{2+1} = \sqrt{2+a_2} = \sqrt{2+\sqrt{2+\sqrt{2}}}$

$n=3$: $a_{3+1} = \sqrt{2+a_3} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \approx 1.993$

$n=4$: $a_{4+1} = \sqrt{2+a_4} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \approx 1.999$

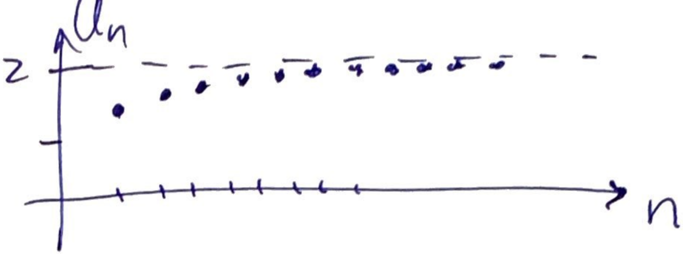
So "it appears" that the

$$\boxed{\lim_{n\to\infty} a_n = 2}$$

$a_1 = \sqrt{2}$
$a_2 = \sqrt{2+\sqrt{2}}$
$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$
$\vdots$

OK. Good. Prove it!

Graph



How do we prove it?

* Proofs: In our College Geometry Class {that is until Calif. AB 705 was passed} we used Geometry to teach logic and the art of the proof.

Types: • Direct Proof

statements	Reason
1. $a \to b$	1. Given
2. $b \to c$	2. Given
3. $a \to c$	3. Transitive

"Nothing but the facts"

• Indirect Proof

If $a \to b$ then not $b \to$ not a

$\boxed{\text{Ex}}$ if water the grass (a), then it is green (b)

Does the other way work? If its green did you water it?

Ans. No

EX cont.

if the grass is NOT green then the grass was not watered.

If $a \to b$ then not $b \to$ not a

In use we assume "not b" occured and then we proceed to find a contradiction so not (not b) holds but not (not b) = b.

- Third Type ~~that~~ deals with patterns

Proof by Induction

EX $\lim_{n \to \infty} \sqrt{2 + a_n} = 2$

(i) Show it works for $n = 1, 2, 3 \ldots$

(ii) Assume it works for n in general

(iii) Prove (show) it works for $n+1$

(iv) Assume therefor it works for all n.

EX It appears that

$$1+3+5+7+\cdots+\underbrace{(2n-1)}_{\text{odd \#}}=n^2$$

Prove it:

Proof by induction:

(i) $n=1$: $1 \stackrel{?}{=} 1^2$ ✓

$n=2$: $1+3 \stackrel{?}{=} 2^2$ ✓

$n=3$: $1+3+5 \stackrel{?}{=} 3^2$ ✓

$\vdots$

(ii) Assume this works for n:

$$1+3+5+\cdots+(2n-1)=n^2$$

(iii) Show it works for $n+1$:

LHS RHS

$$1+3+5+\cdots+(2n-1)+(2(n+1)-1) \stackrel{?}{=} (n+1)^2$$

At this point problems requires diffrent approaches...

Here note that $1+3+5+\cdots+(2n-1)$ is assumed to be n^2

So the LHS becomes

$$\underbrace{1+3+5+\cdots+(2n-1)}_{n^2}+(2(n+1)-1)$$

$$n^2+(2(n+1)-1)$$

$$n^2+2n+2-1$$

$$n^2+2n+1$$

LHS $=(n+1)^2$ which is indeed the RHS also $(n+1)^2$ ✓ QED

(iv) we conclude step (ii) holds for all n.

$\boxed{EX}$ Prove $\underbrace{\overset{a_1}{1}+2+3+\cdots+\overset{a_n}{n}}_{\text{arithmetic so could use}} = \frac{n(n+1)}{2}$

$S_n = \left(\frac{a_1+a_n}{2}\right)\cdot n$

BUT Lets use induction 1st.

(i) $n=1$: $1 \stackrel{?}{=} \frac{1\cdot(1+1)}{2}$ ✓

$n=2$: $1+2 \stackrel{?}{=} \frac{2(2+1)}{2}$ ✓

$n=3$: $1+2+3 \stackrel{?}{=} \frac{3(3+1)}{2}$ ✓

∴ "appears" to hold true

(ii) Assume $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

(iii) Show $\underbrace{1+2+3+\cdots+n}+(n+1) = \frac{(n+1)((n+1)+1)}{2}$

LHS: $= \frac{n(n+1)}{2} + (n+1)$

$= (n+1)\left[\frac{n}{2}+1\right]$

$= (n+1)\left(\frac{n+2}{2}\right)$

$= \frac{(n+1)((n+1)+1)}{2}$ ✓

(iv) Conclusion:

the sum works for all "n".

Q.E.D.

$\boxed{\text{EX}}$ Prove by induction

$$2^3+4^3+6^3+\cdots+(2n)^3 = 2n^2(n+1)^2$$

(i)

$n=1$: $(2\cdot1)^3 \stackrel{?}{=} 2\cdot1^2\cdot(1+1)^2$ ✓

$n=2$: $2^3+(2\cdot2)^3 \stackrel{?}{=} 2\cdot2^2(2+1)^2$

$8+64 \stackrel{?}{=} 2\cdot4\cdot9$ ✓

(ii) Assume $2^3+4^3+\cdots+(2n)^3 = 2n^2(n+1)^2$

(iii) Show $\underbrace{2^3+4^3+\cdots+(2n)^3}+(2(n+1))^3 = 2(n+1)^2((n+1)+1)^2$

$$= \underline{\underline{2(n+1)^2(n+2)^2}}$$

$$\text{LHS} = 2n^2(n+1)^2 + 2^3(n+1)^3$$

$$= 2(n+1)^2\left[\underbrace{\underbrace{n^2+2^2(n+1)}_{n^2+4n+4}}_{(n+2)^2}\right]$$

RHS

match!

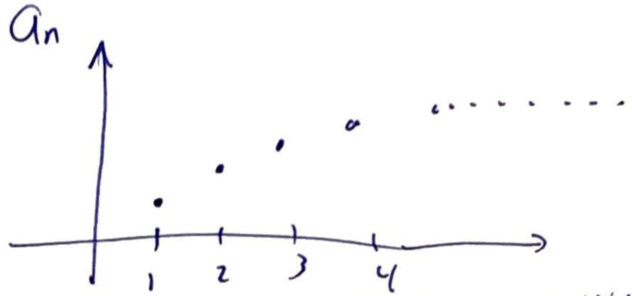
$$\text{LHS} = \underline{\underline{2(n+1)^2(n+2)^2}}$$

(iv) we conclude $2^3+4^3+6^3+\cdots(2n)^3 = 2n^2(n+1)^2$

holds for all n.

* Inequalities

$\boxed{\text{EX}}$ Prove $a_{n+1} = \sqrt{2 + a_n}$, $a_1 = \sqrt{2}$ is monotonic. That is $\boxed{a_{n+1} > a_n}$
increasing



$a_1 = \sqrt{2} \approx 1.1414$

(i) $n=1$: $a_2 = \sqrt{2+\sqrt{2}} \approx 1.847$ $\quad > a_2 > a_1$

$n=2$: $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \approx 1.96$ $\quad > a_3 > a_2$

$\vdots$

it is shown that for $a_1, a_2\ a_3$ we have increasing monotonicity

(ii) Assume $\quad a_{n+1} > a_n$

(iii) Show $\quad a_{n+2} > a_{n+1}$.

Note that if $a_{n+1} > a_n$

- add 2 to both sides $\quad 2 + a_{n+1} > 2 + a_n$
- $\sqrt{\ }$ both sides $\quad \underbrace{\sqrt{2 + a_{n+1}}}_{a_{n+2}} > \underbrace{\sqrt{2 + a_n}}_{a_{n+1}}$

$$\underline{\underline{a_{n+2} > a_{n+1}}}$$

(iv) Conclude therefore that $a_{n+1} > a_n$ holds for all n

QED