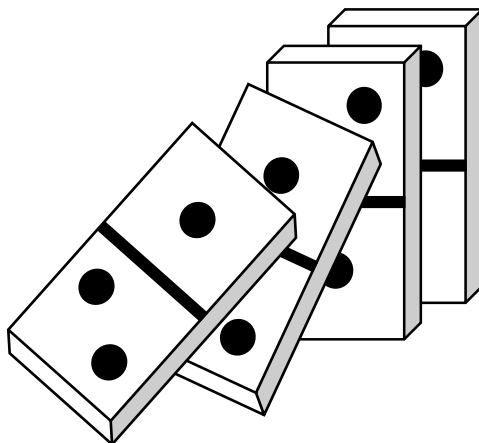

Mathematical Induction

Let's motivate our discussion by considering an example first. What happens when we add the first n positive odd integers? The table below shows what results for the first few values of n .

n	Sum of the first n odd positive integers
1	1
2	$1 + 3 = 4$
3	$1 + 3 + 5 = 9$
4	$1 + 3 + 5 + 7 = 16$
5	$1 + 3 + 5 + 7 + 9 = 25$

If you look closely at the sums we get in each case, you might notice that the result is just n^2 . The question we might ask next is, "Does this happen every time?" In other words, if we take the first n positive odd integers, will their sum always be n^2 ? You may not be surprised to learn that the answer is yes, but how do we know that? Unfortunately, we cannot simply assume based on the results above that it will work for every positive integer (see example 4). Instead, we have to *prove* this result, and one way to do so is to use the principle of **Mathematical Induction**.

Mathematical Induction is a method by which we can prove many formulas, equations, and other mathematical statements whose variables represent positive integers. The idea behind it is similar to the "domino effect." If we want to knock over a set of dominoes standing in a row, we don't have to knock over each one individually. If they're close enough to each other, we simply knock over the first domino. The first domino will then knock over the second, the second will knock over the third, and so on. Ultimately, all of the dominoes will fall.



The principle of Mathematical Induction works the same way. When we want to prove a statement to be true for all positive integer values n , we show that the statement is true when $n = 1$. Then we show that if the statement is true for a positive integer k , then it is true for the next integer, $k + 1$. By doing this, we show that the statement holds when $n = 1$, which means it must also hold when $n = 1 + 1 = 2$, which, in turn, means it holds for $n = 2 + 1 = 3$, and so on. Therefore the statement will hold for all positive integers n . We summarize this principle below.

The Principle of Mathematical Induction

To prove a mathematical statement in terms of the variable n , we show that:

- 1.) The statement is true for $n = 1$.*
- 2.) If the statement is true for $n = k$, then it is true for $n = k + 1$.

*Some statements may not hold for $n = 1$. In fact, some may not work for the first few positive integers. In such cases, the first step will need to be verified for the first integer that makes the statement true.

In practice, here is how the process works. In the first step (sometimes called the **base case**), we simply plug $n = 1$ into the statement and verify that it is true. For the second step, we make the assumption that the statement is true when $n = k$. This is called the **induction hypothesis**. Operating under this assumption, we then verify that the statement must also work when $n = k + 1$. This will involve plugging in $k + 1$ for n and using the induction hypothesis to again verify that the statement holds.

Example 1

For our first example, let's go back to the problem we considered earlier involving the sum of the first n positive odd integers. What we want to prove is that for any positive integer n ,

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

Step 1

First we show that the statement holds when $n = 1$. To do this, we simply plug 1 in for n in the equation above and verify that we get the same thing on both sides:

$$2 \cdot 1 - 1 \stackrel{?}{=} 1^2$$

$$1 = 1 \quad \checkmark$$

Step 2

Next, we assume that the statement is true for $n = k$. In other words, assume that

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$$

Now we must show that it will work for $n = k + 1$. That is, we need to show that

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$$

Notice on the left side of the equation that the first k terms are

$$1 + 3 + 5 + 7 + \cdots + (2k - 1),$$

which we are assuming is equal to k^2 . Thus we can simply substitute and proceed to simplify as follows:

$$\begin{aligned} \underbrace{1 + 3 + 5 + 7 + \cdots + (2k - 1)}_{k^2} + (2(k + 1) - 1) &\stackrel{?}{=} (k + 1)^2 \\ k^2 + (2(k + 1) - 1) &\stackrel{?}{=} (k + 1)^2 \\ k^2 + 2k + 2 - 1 &\stackrel{?}{=} (k + 1)^2 \\ (k^2 + 2k + 1) &\stackrel{?}{=} (k + 1)^2 \\ (k + 1)^2 &= (k + 1)^2 \quad \checkmark \end{aligned}$$

The proof is now complete! ■

Example 2

Use mathematical induction to prove that $2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$.

Step 1

We show the statement is true when $n = 1$.

$$\begin{aligned} 2^1 &\stackrel{?}{=} 2^{1+1} - 2 \\ 2 &\stackrel{?}{=} 2^2 - 2 \\ 2 &\stackrel{?}{=} 4 - 2 \\ 2 &= 2 \quad \checkmark \end{aligned}$$

Step 2

Assume the statement holds for $n = k$. In other words, assume that

$$2 + 2^2 + 2^3 + 2^4 + \cdots + 2^k = 2^{k+1} - 2$$

We now show that

$$2 + 2^2 + 2^3 + 2^4 + \dots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$$

To proceed, we apply the induction hypothesis (the assumption we made above) to the left-hand side of the equation and simplify as follows:

$$\underbrace{2 + 2^2 + 2^3 + 2^4 + \dots + 2^k}_{2^{k+1}-2} + 2^{k+1} \stackrel{?}{=} 2^{(k+1)+1} - 2$$

$$2^{k+1} - 2 + 2^{k+1} \stackrel{?}{=} 2^{k+2} - 2$$

$$2 \cdot 2^{k+1} - 2 \stackrel{?}{=} 2^{k+2} - 2$$

$$2^{1+k+1} - 2 \stackrel{?}{=} 2^{k+2} - 2$$

$$2^{k+2} - 2 = 2^{k+2} - 2 \quad \checkmark$$

Thus, the statement is proven to work for all positive integers. ■

Example 3

Prove that $2^n > 3n$ for $n \geq 4$.

Step 1

Notice the restriction in this case that $n \geq 4$. We can easily verify by plugging in $n = 1, 2$, and 3 that the statement is false for each of those values. This means that our first step is to show the statement is true for $n = 4$. This is easily verifiable.

$$2^4 \stackrel{?}{>} 3 \cdot 4$$

$$16 > 12 \quad \checkmark$$

Step 2

Next, we assume that $2^k > 3k$ for some positive integer $k \geq 4$. We need to show that

$$2^{k+1} > 3(k+1)$$

Since we're assuming that $2^k > 3k$, it follows that $2 \cdot 2^k > 2 \cdot 3k$, which we can simplify to $2^{k+1} > 6k$. From this we have

$$2^{k+1} > 6k = 3k + 3k \geq \underbrace{3k + 12}_{\text{since } k \geq 4} > 3k + 3 = 3(k+1)$$

In short, we have shown that $2^{k+1} > 3(k+1)$, as required. ■

Example 4

We mentioned that showing that the statement $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ works for $n = 1, 2, 3$, and 4 is not sufficient enough to prove that it works for all positive integers. Simply plugging in a few values like this does not verify that the equation will always work, hence the need for a proof by mathematical induction.

To further see the dangers of making an assumption about the validity of a statement based upon plugging in a few values, consider the following. Let's plug the first few positive integers into the polynomial $n^2 + n + 11$.

n	$n^2 + n + 11$
1	$1^2 + 1 + 11 = 13$
2	$2^2 + 2 + 11 = 17$
3	$3^2 + 3 + 11 = 23$
4	$4^2 + 4 + 11 = 31$

Notice that the sum in each of these cases is a prime number. If we continue to plug in positive integer values for n , will we continue to get a prime number result? Let's try a few more and see.

n	$n^2 + n + 11$
5	$5^2 + 5 + 11 = 41$
6	$6^2 + 6 + 11 = 53$
7	$7^2 + 7 + 11 = 67$
8	$8^2 + 8 + 11 = 83$
9	$9^2 + 9 + 11 = 101$

Notice that each result is still a prime number. At this point, you might be tempted to say that the results will always be prime. However, if we plug in $n = 10$, we get

$$10^2 + 10 + 11 = 121,$$

which is *not* prime! This shows that we cannot "prove" a statement by simply plugging in a few values, nor should we assume that a statement is valid simply because it seems to be, based on a few results. ■

Exercises

In exercises 1 – 5, prove that the given statement holds for all positive integers n .

$$1. \quad 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$$

$$2. \quad 5 + 9 + 13 + 17 + \cdots + (4n+1) = n(2n+3)$$

$$3. \quad 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$4. \quad 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$5. \quad 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}$$

In exercises 6 – 8, prove that the given inequality holds for the specified integer values of n .

$$6. \quad 3^n > 4n, \quad n \geq 2$$

$$7. \quad 2^n > (n+1)^2, \quad n \geq 6$$

$$8. \quad n^3 > (n+1)^2, \quad n \geq 3$$