6.1 Inner Products

DEFINITION 1 An ***inner product*** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a ***real inner product space***.

DEFINITION 2 If V is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

THEOREM 6.1.1 *If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, and if k is a scalar, then:*

(a) $\|\mathbf{v}\| \geq 0$ *with equality if and only if* $\mathbf{v} = \mathbf{0}$.

(b) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

(d) $d(\mathbf{u}, \mathbf{v}) \geq 0$ *with equality if and only if* $\mathbf{u} = \mathbf{v}$.

$$w_1, w_2, \ldots, w_n$$

are *positive* real numbers, which we will call **weights**, and if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n \tag{2}$$

defines an inner product on R^n that we call the ***weighted Euclidean inner product with weights*** $w_1, w_2, \ldots, w_n$.

DEFINITION 3 If V is an inner product space, then the set of points in V that satisfy

$$\|\mathbf{u}\| = 1$$

is called the ***unit sphere*** or sometimes the ***unit circle*** in V.

matrix inner products: if u · v is the Euclidean inner product on R^n, then the formula $\langle u, v\rangle = Au \cdot Av$ also defines an inner product. $\langle u, v\rangle = (Av)^T Au = v^T A^T Au$

If u = U and v = V are matrices in the vector space Mnxn, then the formula $\langle u, v\rangle = tr(U^T V)$ defines an inner product on Mnxn called the ***standard inner product*** on that space

If are polynomials in P_n, then the following formula defines an inner product on P_n (verify) that we will call the *standard inner product* on this space: $\langle p,q\rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$

EXAMPLE 10 An Integral Inner Product on $C[a, b]$

Let $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ be two functions in $C[a, b]$ and define

$$\langle \mathbf{f}, \mathbf{g}\rangle = \int_a^b f(x)g(x)\,dx \quad (12)$$

We will show that this formula defines an inner product on $C[a, b]$ by verifying the four inner product axioms for functions $\mathbf{f} = f(x)$, $\mathbf{g} = g(x)$, and $\mathbf{h} = h(x)$ in $C[a, b]$:

Axiom 1: $\langle \mathbf{f}, \mathbf{g}\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle \mathbf{g}, \mathbf{f}\rangle$

Axiom 2:
$$\begin{aligned}\langle \mathbf{f}+\mathbf{g}, \mathbf{h}\rangle &= \int_a^b (f(x)+g(x))h(x)\,dx \\ &= \int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx \\ &= \langle \mathbf{f}, \mathbf{h}\rangle + \langle \mathbf{g}, \mathbf{h}\rangle\end{aligned}$$

Axiom 3: $\langle k\mathbf{f}, \mathbf{g}\rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\langle \mathbf{f}, \mathbf{g}\rangle$

Axiom 4: If $\mathbf{f} = f(x)$ is any function in $C[a, b]$, then

$$\langle \mathbf{f}, \mathbf{f}\rangle = \int_a^b f^2(x)\,dx \geq 0 \quad (13)$$

since $f^2(x) \geq 0$ for all x in the interval $[a, b]$. Moreover, because f is continuous on $[a, b]$, the equality in Formula (13) holds if and only if the function f is identically zero on $[a, b]$, that is, if and only if $\mathbf{f} = \mathbf{0}$; and this proves that Axiom 4 holds.

THEOREM 6.1.2 *If* **u**, **v**, *and* **w** *are vectors in a real inner product space* V, *and if* k *is a scalar, then:*

(*a*) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(*b*) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(*c*) $\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$

(*d*) $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$

(*e*) $k\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle$

6.2 Angle and Orthogonality in Inner Product Spaces

Recall from Formula (20) of Section 3.2 that the angle θ between two vectors **u** and **v** in R^n is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) \tag{1}$$

We were assured that this formula was valid because it followed from the Cauchy–Schwarz inequality (Theorem 3.2.4) that

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1 \tag{2}$$

as required for the inverse cosine to be defined. The following generalization of the Cauchy–Schwarz inequality will enable us to define the angle between two vectors in *any* real inner product space.

THEOREM 6.2.1 Cauchy–Schwarz Inequality

If **u** *and* **v** *are vectors in a real inner product space* V, *then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\| \tag{3}$$

(Figure 6.2.1). This enables us to *define* the ***angle*** $\boldsymbol{\theta}$ ***between*** **u** ***and*** **v** to be

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

THEOREM 6.2.2 *If* **u**, **v**, *and* **w** *are vectors in a real inner product space* V, *and if* k *is any scalar, then:*

(*a*) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(*b*) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ **[Triangle inequality for distances]**

DEFINITION 1 Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V called ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

THEOREM 6.2.3 Generalized Theorem of Pythagoras

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in a real inner product space, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

DEFINITION 2 If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^\perp$.

THEOREM 6.2.4 *If W is a subspace of a real inner product space V, then:*

(*a*) $W^\perp$ *is a subspace of* V.

(*b*) $W \cap W^\perp = \{\mathbf{0}\}$.

EXAMPLE 6 Basis for an Orthogonal Complement

Let W be the subspace of R^6 spanned by the vectors

$$\mathbf{w}_1 = (1, 3, -2, 0, 2, 0), \quad \mathbf{w}_2 = (2, 6, -5, -2, 4, -3),$$
$$\mathbf{w}_3 = (0, 0, 5, 10, 0, 15), \quad \mathbf{w}_4 = (2, 6, 0, 8, 4, 18)$$

Find a basis for the orthogonal complement of W.

Solution The subspace W is the same as the row space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Since the row space and null space of A are orthogonal complements, our problem reduces to finding a basis for the null space of this matrix. In Example 4 of Section 4.7 we showed that

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for this null space. Expressing these vectors in comma-delimited form (to match that of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$), we obtain the basis vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

You may want to check that these vectors are orthogonal to $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ by computing the necessary dot products. ◄

6.3 Gram–Schmidt Process; QR-Decomposition

DEFINITION 1 A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

THEOREM 6.3.1 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.*

THEOREM 6.3.2

(*a*) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if* **u** *is any vector in V, then*

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \tag{3}$$

(*b*) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if* **u** *is any vector in V, then*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n \tag{4}$$

Using the terminology and notation from Definition 2 of Section 4.4, it follows from Theorem 6.3.2 that the coordinate vector of a vector **u** in V relative to an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

$$(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}, \ldots, \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \right) \tag{6}$$

and relative to an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

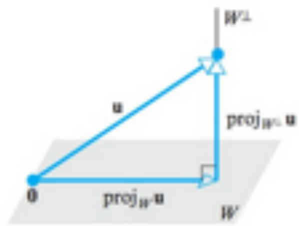
$$(\mathbf{u})_S = (\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle) \tag{7}$$

THEOREM 6.3.3 Projection Theorem

If W is a finite-dimensional subspace of an inner product space V, then every vector **u** *in V can be expressed in exactly one way as*

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 \tag{8}$$

where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$.



▲ Figure 6.3.1

These are called the ***orthogonal projection of* u *on* W** and the ***orthogonal projection of* u *on* $W^\perp$**, respectively. The vector $\mathbf{w}_2$ is also called the ***component of* u *orthogonal to* W**. Using the notation in (9), Formula (8) can be expressed as

$$\mathbf{u} = \text{proj}_W \mathbf{u} + \text{proj}_{W^\perp} \mathbf{u} \tag{10}$$

(Figure 6.3.1). Moreover, since $\text{proj}_{W^\perp}\mathbf{u} = \mathbf{u} - \text{proj}_W \mathbf{u}$, we can also express Formula (10) as

$$\mathbf{u} = \text{proj}_W \mathbf{u} + (\mathbf{u} - \text{proj}_W \mathbf{u}) \tag{11}$$

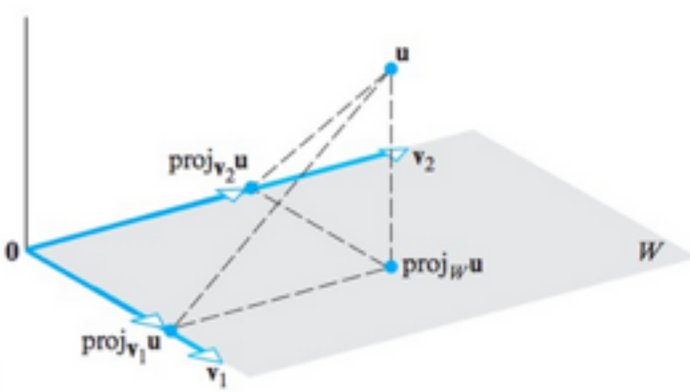
THEOREM 6.3.4 *Let W be a finite-dimensional subspace of an inner product space V.*

(a) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r\rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \tag{12}$$

(b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r\rangle\mathbf{v}_r \tag{13}$$



▶ Figure 6.3.2

THEOREM 6.3.5 *Every nonzero finite-dimensional inner product space has an orthonormal basis.*

The Gram–Schmidt Process

To convert a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computations:

Step 1. $\mathbf{v}_1 = \mathbf{u}_1$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

Step 4. $\mathbf{v}_4 = \mathbf{u}_4 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_3\rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$\vdots$

(continue for r steps)

Optional Step. To convert the orthogonal basis into an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_r\}$, normalize the orthogonal basis vectors.

Assume that the vector space R^3 has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = (1, 1, 1), \quad \mathbf{u}_2 = (0, 1, 1), \quad \mathbf{u}_3 = (0, 0, 1)$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

Solution

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

$$= (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Thus,

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

THEOREM 6.3.6 *If W is a finite-dimensional inner product space, then:*

(*a*) *Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.*

(*b*) *Every orthonormal set in W can be enlarged to an orthonormal basis for W.*

THEOREM 6.3.7 *QR*-Decomposition

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

It follows from Theorem 6.3.7 that every invertible matrix has a QR-decomposition.

Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt process with normalization to these column vectors yields the orthonormal vectors (see Example 8)

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, it follows from Formula (16) that R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

from which it follows that a QR-decomposition of A is

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{=}{=} \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}} \blacktriangleleft$$