5.1 Eigenvalues and Eigenvectors

DEFINITION 1 If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an ***eigenvector*** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of A (or of T_A), and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** $\boldsymbol{\lambda}$.

THEOREM 5.1.1 *If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation*

$$\det(\lambda I - A) = 0 \tag{1}$$

This is called the ***characteristic equation*** *of A.*

THEOREM 5.1.2 *If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.*

THEOREM 5.1.3 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(*a*) *λ is an eigenvalue of A.*

(*b*) *λ is a solution of the characteristic equation* $\det(\lambda I - A) = 0$.

(*c*) *The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.*

(*d*) *There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.*

Notice that x = 0 is in every eigenspace but is not an eigen- vector (see Definition 1). In the exercises we will ask you to show that this is the only vector that distinct eigenspaces have in common.

THEOREM 5.1.4 *A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.*

THEOREM 5.1.5 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) *The column vectors of A are distinct and linearly independent.*

(*i*) *The row vectors of A are distinct and linearly independent.*

(*j*) *The column vectors of A span R^n.*

(*k*) *The row vectors of A span R^n.*

(*l*) *The column vectors of A form a basis for R^n.*

(*m*) *The row vectors of A form a basis for R^n.*

(*n*) *A has rank n.*

(*o*) *A has nullity* 0.

(*p*) *The orthogonal complement of the null space of A is R^n.*

(*q*) *The orthogonal complement of the row space of A is* $\{\mathbf{0}\}$.

(*r*) *The kernel of T_A is* $\{\mathbf{0}\}$.

(*s*) *The range of T_A is R^n.*

(*t*) *T_A is one-to-one.*

(*u*) $\lambda = 0$ *is not an eigenvalue of A.*

DEFINITION 2 If $T: V \rightarrow V$ is a linear operator on a vector space V, then a nonzero vector $\mathbf{x}$ in V is called an ***eigenvector*** of T if $T(\mathbf{x})$ is a scalar multiple of $\mathbf{x}$; that is,

$$T(\mathbf{x}) = \lambda \mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of T, and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** $\boldsymbol{\lambda}$.

If $D: C^\infty \to C^\infty$ is the differentiation operator on the vector space of functions with continuous derivatives of all orders on the interval $(-\infty, \infty)$, and if λ is a constant, then

$$D(e^{\lambda x}) = \lambda e^{\lambda x}$$

so that λ is an eigenvalue of D and $e^{\lambda x}$ is a corresponding eigenvector. ◀

In vector spaces of functions eigenvectors are commonly re- ferred to as eigenfunctions

The eigenvectors that we have been studying are sometimes called right eigenvectors to distinguish them from left eigen- vectors, which are n × 1 column matrices x that satisfy the equation xTA = μxT for some scalar μ.

5.2 Diagonalization

Consider A and P, n × n matrices, and P is invertible, such that

$$A \to P-1AP$$

the matrix A is mapped into the matrix $P^{-1}AP$ are called **similarity transformations**.

If we let B = P −1AP , then A and B have the same determinant:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= \frac{1}{\det(P)}\det(A)\det(P) = \det(A)$$

Any property that is preserved by a similarity transformation is called a **similarity invariant** and is said to be ***invariant under similarity***.

Table 1 Similarity Invariants

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues.
Eigenspace dimension	If λ is an eigenvalue of A (and hence of $P^{-1}AP$) then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

DEFINITION 1 If A and B are square matrices, then we say that ***B is similar to A*** if there is an invertible matrix P such that $B = P^{-1}AP$.

DEFINITION 2 A square matrix A is said to be ***diagonalizable*** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to ***diagonalize*** A.

THEOREM 5.2.1 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(*a*) *A is diagonalizable.*

(*b*) *A has n linearly independent eigenvectors.*

THEOREM 5.2.2

(*a*) *If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix A, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.*

(*b*) *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

A Procedure for Diagonalizing an $n \times n$ Matrix

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$ whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of P.

If you are concerned only in determining whether a matrix is diagonalizable and not with actually finding a diagonalizing matrix P , then it is not necessary to compute bases for the eigenspaces — it suffices to find the dimensions of the eigenspaces. if it has n distinct eigenvalues.

THEOREM 5.2.3 *If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.*

The problem of computing powers of a matrix is greatly simplified when the matrix is diagonalizable.

$(P^{-1}AP)2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P$ from which we obtain the relationship $P^{-1}A^2P = D^2$. More generally

$$P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

$$A^k = P\,D^k\,P^{-1}$$

Theorem 5.2.2(b) does not completely settle the diagonalizability question since it only guarantees that a square matrix with n distinct eigenvalues is diagonalizable; it does not preclude the possibility that there may exist diagonalizable matrices with fewer than n distinct eigenvalues.

If λ_0 is an eigenvalue of A, then the dimension of the eigenspace corresponding to λ_0 cannot exceed the multiplicity of λ_0 as a factor of the characteristic polynomial of A.

If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the **geometric multiplicity** of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity** of λ_0 .

THEOREM 5.2.4 Geometric and Algebraic Multiplicity

If A is a square matrix, then:

(*a*) *For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.*

(*b*) *A is diagonalizable if and only if the characteristic polynomial of A is factorable into linear terms and the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.*