Vector Spaces

4.1 Vector space axioms

DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

THEOREM 4.1.1 *Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:*

(*a*) $0\mathbf{u}=\mathbf{0}$

(*b*) $k\mathbf{0}=\mathbf{0}$

(*c*) $(-1)\mathbf{u}=-\mathbf{u}$

(*d*) *If* $k\mathbf{u}=\mathbf{0}$, *then* $k=0$ *or* $\mathbf{u}=\mathbf{0}$.

Vector Spaces

4.2 Subspaces

DEFINITION 1 A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

THEOREM 4.2.1 *If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.*

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

THEOREM 4.2.2 *If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.*

DEFINITION 2 If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{2}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

THEOREM 4.2.3 *If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:*

(a) The set W of all possible linear combinations of the vectors in S is a subspace of V.

(b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

DEFINITION 3 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V ***generated*** by S, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ ***span*** W. We denote this subspace as

$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S)$$

THEOREM 4.2.4 *The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.*

THEOREM 4.2.5 *If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A: R^n \to R^m$ is a subspace of R^n.*

THEOREM 4.2.6 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then*

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

4.3 Linear Dependence and Independence

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***.

THEOREM 4.3.1 *A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

THEOREM 4.3.2

(*a*) *A finite set that contains $\mathbf{0}$ is linearly dependent.*

(*b*) *A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.*

(*c*) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

THEOREM 4.3.3 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.*

DEFINITION 2 If $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \ldots, \mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$.

THEOREM 4.3.4 *If the functions $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ have $n-1$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty, \infty)$.*

4.4 Coordinates and Basis

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a ***basis*** for V if:

(a) S spans V.

(b) S is linearly independent.

THEOREM 4.4.1 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

Vector Spaces

DEFINITION 2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the ***coordinates* of v relative to the basis *S***. The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the ***coordinate vector of* v *relative to S***; it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n) \tag{6}$$

4.5 Dimension

THEOREM 4.5.1 *All bases for a finite-dimensional vector space have the same number of vectors.*

THEOREM 4.5.2 *Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis.*

(a) If a set in V has more than n vectors, then it is linearly dependent.

(b) If a set in V has fewer than n vectors, then it does not span V.

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

THEOREM 4.5.3 Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(a) If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.

(b) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\})$$

THEOREM 4.5.4 *Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.*

THEOREM 4.5.5 *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(*a*) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(*b*) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

THEOREM 4.5.6 *If W is a subspace of a finite-dimensional vector space V, then:*

(*a*) *W is finite-dimensional.*

(*b*) $\dim(W) \leq \dim(V)$.

(*c*) *$W = V$ if and only if* $\dim(W) = \dim(V)$.

4.6 Change of Basis

The Change-of-Basis Problem If $\mathbf{v}$ is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

Solution of the Change-of-Basis Problem If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$, then for each vector $\mathbf{v}$ in V, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} \tag{7}$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are

$$[\mathbf{u}'_1]_B, \quad [\mathbf{u}'_2]_B, \ldots, \quad [\mathbf{u}'_n]_B \tag{8}$$

The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

$$[\mathbf{v}]_B = P_{B' \to B}[\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = P_{B \to B'}[\mathbf{v}]_B$$

THEOREM 4.6.1 *If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.*

A Procedure for Computing Transition Matrices

Step 1. Form the partitioned matrix [new basis | old basis] in which the basis vectors are in column form.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be [I | transition matrix from old to new] where I is an identity matrix.

Step 4. Extract the matrix on the right side of the matrix obtained in Step 3.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}]$$

THEOREM 4.6.2 *Let $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be any basis for the vector space R^n and let $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for R^n. If the vectors in these bases are written in column form, then*

$$P_{B' \to S} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n] \tag{15}$$

4.7 Row Space, Column Space, and Null Space

DEFINITION 1 For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ &\vdots \\ \mathbf{r}_m &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

in R^n that are formed from the rows of A are called the ***row vectors*** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the ***column vectors*** of A.

DEFINITION 2 If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the ***row space*** of A, and the subspace of R^m spanned by the column vectors of A is called the ***column space*** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the ***null space*** of A.

The vector x_0 in Formula (3) is called a *particular solution* of Ax = b, and the remaining part of the formula is called the *general solution* of Ax = 0.

THEOREM 4.7.1 *A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.*

THEOREM 4.7.2 *If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{3}$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

The general solution of a consistent linear system can be expressed as the sum of a particular solution of that system and the general solution of the corresponding homogeneous system.

THEOREM 4.7.3 *Elementary row operations do not change the null space of a matrix.*

THEOREM 4.7.4 *Elementary row operations do not change the row space of a matrix.*

THEOREM 4.7.5 *If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.*

THEOREM 4.7.6 *If A and B are row equivalent matrices, then:*

(*a*) *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*

(*b*) *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

Problem Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in R^n, find a subset of these vectors that forms a basis for span(S), and express each vector that is not in that basis as a linear combination of the basis vectors.

Basis for the Space Spanned by a Set of Vectors

Step 1. Form the matrix A whose columns are the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Step 2. Reduce the matrix A to reduced row echelon form R.

Step 3. Denote the column vectors of R by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$.

Step 4. Identify the columns of R that contain the leading 1's. The corresponding column vectors of A form a basis for span(S).

This completes the first part of the problem.

Step 5. Obtain a set of dependency equations for the column vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ of R by successively expressing each $\mathbf{w}_i$ that does not contain a leading 1 of R as a linear combination of predecessors that do.

Step 6. In each dependency equation obtained in Step 5, replace the vector $\mathbf{w}_i$ by the vector $\mathbf{v}_i$ for $i = 1, 2, \ldots, k$.

This completes the second part of the problem.

4.8 Rank, Nullity, and the Fundamental Matrix Spaces

THEOREM 4.8.1 *The row space and the column space of a matrix A have the same dimension.*

Thus, if R is any row echelon form of A, it must be true that:

dim(row space of A) = dim(row space of R)
dim(column space of A) = dim(column space of R)

DEFINITION 1 The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by rank(A); the dimension of the null space of A is called the ***nullity*** of A and is denoted by nullity(A).

THEOREM 4.8.2 Dimension Theorem for Matrices
If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n \tag{4}$$

Proof Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These fall into two distinct categories: the leading variables and the free variables. Thus,

$$\begin{bmatrix}\text{number of leading}\\ \text{variables}\end{bmatrix} + \begin{bmatrix}\text{number of free}\\ \text{variables}\end{bmatrix} = n$$

But the number of leading variables is the same as the number of leading 1's in any row echelon form of A, which is the same as the dimension of the row space of A, which is the same as the rank of A. Also, the number of free variables in the general solution of $A\mathbf{x} = \mathbf{0}$ is the same as the number of parameters in that solution, which is the same as the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, which is the same as the nullity of A.

THEOREM 4.8.3 *If A is an $m \times n$ matrix, then*

(*a*) *rank*(A) = *the number of leading variables in the general solution of* $A\mathbf{x} = \mathbf{0}$.

(*b*) *nullity*(A) = *the number of parameters in the general solution of* $A\mathbf{x} = \mathbf{0}$.

THEOREM 4.8.4 *If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.*

There are six important vector spaces associated with a matrix A and its transpose A^T:

row space of A	row space of A^T
column space of A	column space of A^T
null space of A	null space of A^T

THEOREM 4.8.5 *If A is any matrix, then rank(A) = rank(A^T).*

$$\text{rank}(A) + \text{nullity}(A^T) = m \tag{5}$$

This alternative form of Formula (4) makes it possible to express the dimensions of all four fundamental spaces in terms of the size and rank of A. Specifically, if rank$(A) = r$, then

$$\begin{array}{ll} \dim[\text{row}(A)] = r & \dim[\text{col}(A)] = r \\ \dim[\text{null}(A)] = n - r & \dim[\text{null}(A^T)] = m - r \end{array} \tag{6}$$

DEFINITION 2 If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^\perp$.

THEOREM 4.8.6 *If W is a subspace of R^n, then:*

(a) $W^\perp$ *is a subspace of* R^n.

(b) *The only vector common to* W *and* $W^\perp$ *is* $\mathbf{0}$.

(c) *The orthogonal complement of* $W^\perp$ *is* W.

THEOREM 4.8.7 *If A is an $m \times n$ matrix, then:*

(*a*) *The null space of A and the row space of A are orthogonal complements in R^n.*

(*b*) *The null space of A^T and the column space of A are orthogonal complements in R^m.*

THEOREM 4.8.8 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) *The column vectors of A are distinct and linearly independent.*

(*i*) *The row vectors of A are distinct and linearly independent.*

(*j*) *The column vectors of A span R^n.*

(*k*) *The row vectors of A span R^n.*

(*l*) *The column vectors of A form a basis for R^n.*

(*m*) *The row vectors of A form a basis for R^n.*

(*n*) *A has rank n.*

(*o*) *A has nullity 0.*

(*p*) *The orthogonal complement of the null space of A is R^n.*

(*q*) *The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.*

THEOREM 4.8.9 *Let A be an $m \times n$ matrix.*

(*a*) (***Overdetermined Case***). *If $m > n$, then the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in R^n.*

(*b*) (***Underdetermined Case***). *If $m < n$, then for each vector $\mathbf{b}$ in R^m the linear system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.*