3.1 Vectors in 2-Space, 3-Space, and n-Space

DEFINITION 3 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \tag{10}$$

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n) \tag{11}$$

$$-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n) \tag{12}$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n) \tag{13}$$

THEOREM 3.1.1 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then:*

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(*e*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*f*) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(*g*) $k(m\mathbf{u}) = (km)\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

THEOREM 3.1.2 *If* $\mathbf{v}$ *is a vector in* R^n *and* k *is a scalar, then:*

(*a*) $0\mathbf{v} = \mathbf{0}$

(*b*) $k\mathbf{0} = \mathbf{0}$

(*c*) $(-1)\mathbf{v} = -\mathbf{v}$

DEFINITION 4 If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{14}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination. In the case where $r = 1$, Formula (14) becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

3.2 Norm of a Vector Norm, Dot Product, and Distance in Rn

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$

THEOREM 3.2.1 *If* $\mathbf{v}$ *is a vector in* R^n, *and if* k *is any scalar, then:*

(*a*) $\|\mathbf{v}\| \geq 0$

(*b*) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(*c*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

DEFINITION 2 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \tag{17}$$

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u}\cdot\mathbf{v})=(k\mathbf{u})\cdot\mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v}\cdot\mathbf{v}\geq 0$ *and* $\mathbf{v}\cdot\mathbf{v}=0$ *if and only if* $\mathbf{v}=\mathbf{0}$ **[Positivity property]**

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ *and* $\mathbf{v}=(v_1,v_2,\ldots,v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u}\cdot\mathbf{v}|\leq\|\mathbf{u}\|\|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1v_1+u_2v_2+\cdots+u_nv_n|\leq(u_1^2+u_2^2+\cdots+u_n^2)^{1/2}(v_1^2+v_2^2+\cdots+v_n^2)^{1/2} \tag{23}$$

THEOREM 3.2.6 Parallelogram Equation for Vectors

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n, *then*

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=2\left(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2\right) \tag{24}$$

THEOREM 3.2.7 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n *with the Euclidean inner product, then*

$$\mathbf{u}\cdot\mathbf{v}=\tfrac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\tfrac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 \tag{25}$$

Form	Dot Product	Example	
u a column matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
u a row matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}=\mathbf{v}^T\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
u a column matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\mathbf{u}=\mathbf{u}^T\mathbf{v}^T$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{v}\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$ $\mathbf{u}^T\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$
u a row matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}^T=\mathbf{v}\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{u}\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

If A is an $n \times n$ matrix and **u** and **v** are $n \times 1$ matrices, then it follows from the first row in Table 1 and properties of the transpose that

$$A\mathbf{u}\cdot\mathbf{v}=\mathbf{v}^T(A\mathbf{u})=(\mathbf{v}^TA)\mathbf{u}=(A^T\mathbf{v})^T\mathbf{u}=\mathbf{u}\cdot A^T\mathbf{v}$$
$$\mathbf{u}\cdot A\mathbf{v}=(A\mathbf{v})^T\mathbf{u}=(\mathbf{v}^TA^T)\mathbf{u}=\mathbf{v}^T(A^T\mathbf{u})=A^T\mathbf{u}\cdot\mathbf{v}$$

The resulting formulas

$$A\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot A^T\mathbf{v} \tag{26}$$

$$\mathbf{u}\cdot A\mathbf{v}=A^T\mathbf{u}\cdot\mathbf{v} \tag{27}$$

3.3 Orthogonality

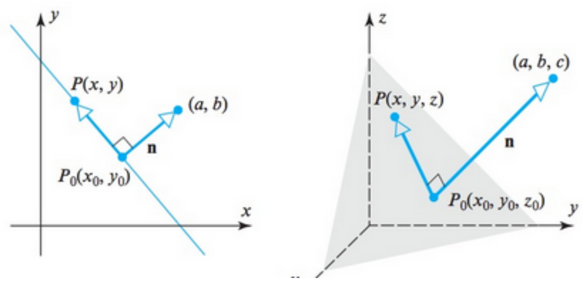
DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

Thus, Equation (1) can be written as

$$a(x - x_0) + b(y - y_0) = 0 \quad \textbf{[line]}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \textbf{[plane]}$$

These are called the ***point-normal*** equations of the line and plane.



THEOREM 3.3.1

(a) *If a and b are constants that are not both zero, then an equation of the form*

$$ax + by + c = 0 \tag{4}$$

represents a line in R^2 with normal $\mathbf{n} = (a, b)$.

(b) *If a, b, and c are constants that are not all zero, then an equation of the form*

$$ax + by + cz + d = 0 \tag{5}$$

represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.

Recall that

$$ax + by = 0 \quad \text{and} \quad ax + by + cz = 0$$

are called *homogeneous equations*. Example 3 illustrates that homogeneous equations in two or three unknowns can be written in the vector form

$$\mathbf{n} \cdot \mathbf{x} = 0 \tag{6}$$

where $\mathbf{n}$ is the vector of coefficients and $\mathbf{x}$ is the vector of unknowns. In R^2 this is called the ***vector form of a line*** through the origin, and in R^3 it is called the ***vector form of a plane*** through the origin.

THEOREM 3.3.2 Projection Theorem

If $\mathbf{u}$ *and* $\mathbf{a}$ *are vectors in* R^n, *and if* $\mathbf{a} \neq 0$, *then* $\mathbf{u}$ *can be expressed in exactly one way in the form* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, *where* $\mathbf{w}_1$ *is a scalar multiple of* $\mathbf{a}$ *and* $\mathbf{w}_2$ *is orthogonal to* $\mathbf{a}$.

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \text{(vector component of } \mathbf{u} \text{ along } \mathbf{a})$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad \text{(vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{a})$$

THEOREM 3.3.3 Theorem of Pythagoras in R^n

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal vectors in* R^n *with the Euclidean inner product, then*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \tag{14}$$

THEOREM 3.3.4

(*a*) *In* R^2 *the distance* D *between the point* $P_0(x_0, y_0)$ *and the line* $ax + by + c = 0$ *is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{15}$$

(*b*) *In* R^3 *the distance* D *between the point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{16}$$

3.4 The Geometry of Linear Systems

THEOREM 3.4.1 *Let* L *be the line in* R^2 *or* R^3 *that contains the point* $\mathbf{x}_0$ *and is parallel to the nonzero vector* $\mathbf{v}$. *Then the equation of the line through* $\mathbf{x}_0$ *that is parallel to* $\mathbf{v}$ *is*

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{1}$$

If $\mathbf{x}_0 = \mathbf{0}$, *then the line passes through the origin and the equation has the form*

$$\mathbf{x} = t\mathbf{v} \tag{2}$$

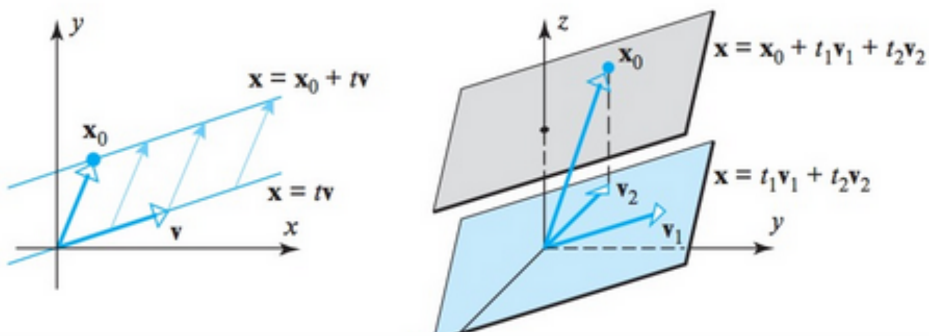
THEOREM 3.4.2 *Let W be the plane in R^3 that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by*

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{3}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{4}$$

Remark Observe that the line through $\mathbf{x}_0$ represented by Equation (1) is the translation by $\mathbf{x}_0$ of the line through the origin represented by Equation (2) and that the plane through $\mathbf{x}_0$ represented by Equation (3) is the translation by $\mathbf{x}_0$ of the plane through the origin represented by Equation (4) (Figure 3.4.4).



DEFINITION 1 If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{5}$$

defines the ***line through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to ***pass through the origin***.

DEFINITION 2 If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{6}$$

defines the ***plane through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}_1$ ***and*** $\mathbf{v}_2$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to ***pass through the origin***.

If $\mathbf{x}_0$ and $\mathbf{x}_1$ are distinct points in R^n, then the line determined by these points is parallel to the vector $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$ (Figure 3.4.5), so it follows from (5) that the line can be expressed in vector form as

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \tag{9}$$

or, equivalently, as

$$\mathbf{x} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1 \tag{10}$$

These are called the ***two-point vector equations*** of a line in R^n.

DEFINITION 3 If $\mathbf{x}_0$ and $\mathbf{x}_1$ are vectors in R^n, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \quad (0 \le t \le 1) \tag{13}$$

defines the ***line segment from*** $\mathbf{x}_0$ ***to*** $\mathbf{x}_1$. When convenient, Equation (13) can be written as

$$\mathbf{x} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1 \quad (0 \le t \le 1) \tag{14}$$

THEOREM 3.4.3 *If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.*

To motivate the result we are seeking, let us compare the solutions of the corresponding linear systems

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

We showed in Examples 5 and 6 of Section 1.2 that the general solutions of these linear systems can be written in parametric form as

homogeneous $\longrightarrow x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$

nonhomogeneous $\longrightarrow x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$

which we can then rewrite in vector form as

homogeneous $\longrightarrow (x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$

nonhomogeneous $\longrightarrow (x_1, x_2, x_3, x_4, x_5, x_6) = \left(-3r - 4s - 2t, r, -2s, s, t, \frac{1}{3}\right)$

By splitting the vectors on the right apart and collecting terms with like parameters, we can rewrite these equations as

homogeneous $\longrightarrow (x_1, x_2, x_3, x_4, x_5) = r(-3, 1, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$ (20)

nonhomogeneous $\longrightarrow (x_1, x_2, x_3, x_4, x_5) = r(-3, 1, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0)$
$+ t(-2, 0, 0, 0, 1, 0) + \left(0, 0, 0, 0, 0, \frac{1}{3}\right)$ (21)

THEOREM 3.4.4 *The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.*

3.5 Cross Product of Vectors

DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{1}$$

THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* 3*-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$]

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$]

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ [*Lagrange's identity*]

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ [*vector triple product*]

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ [*vector triple product*]

THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in* 3*-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

WARNING As suggested by parts (*d*) and (*e*) of Theorem 3.5.1, it is not true in general that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. For example,

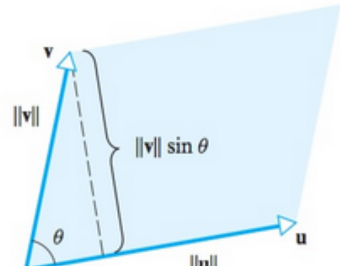
$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}$$

and

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

so

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$$



$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$$

THEOREM 3.5.3 Area of a Parallelogram

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* 3*-space, then* $\|\mathbf{u} \times \mathbf{v}\|$ *is equal to the area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$.

DEFINITION 2 If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in 3-space, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the ***scalar triple product*** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

The scalar triple product of $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ can be calculated from the formula

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \qquad (7)$$

THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

$$\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2*-space determined by the vectors* $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. (*See Figure* 3.5.7*a.*)

(*b*) *The absolute value of the determinant*

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is equal to the volume of the parallelepiped in 3*-space determined by the vectors* $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. (*See Figure* 3.5.7*b.*)

$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 0$ if and only if the vectors u, v, and w lie in the same plane.