CH02 THEOREMS Linear Algebra Anton, 10th ed.

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**DEFINITION 1** If $A$ is a square matrix, then the ***minor of entry*** $a_{ij}$ is denoted by $M_{ij}$ and is defined to be the determinant of the submatrix that remains after the $i$th row and $j$th column are deleted from $A$. The number $(-1)^{i+j} M_{ij}$ is denoted by $C_{ij}$ and is called the ***cofactor of entry*** $a_{ij}$.

**THEOREM 2.1.1** *If $A$ is an $n \times n$ matrix, then regardless of which row or column of $A$ is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.*

**DEFINITION 2** If $A$ is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of $A$ by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{7}$$

[cofactor expansion along the $j$th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{8}$$

[cofactor expansion along the $i$th row]

**THEOREM 2.1.2** *If $A$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22} \cdots a_{nn}$.*

**THEOREM 2.2.1** *Let $A$ be a square matrix. If $A$ has a row of zeros or a column of zeros, then* $\det(A) = 0$.

**THEOREM 2.2.2** *Let $A$ be a square matrix. Then* $\det(A) = \det(A^T)$.

| Relationship | Operation |
|---|---|
| $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = k\det(A)$ | In the matrix $B$ the first row of $A$ was multiplied by $k$. |
| $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = -\det(A)$ | In the matrix $B$ the first and second rows of $A$ were interchanged. |
| $\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ <br> $\det(B) = \det(A)$ | In the matrix $B$ a multiple of the second row of $A$ was added to the first row. |

**THEOREM 2.2.3** *Let $A$ be an $n \times n$ matrix.*

(*a*) *If $B$ is the matrix that results when a single row or single column of $A$ is multiplied by a scalar $k$, then* $\det(B) = k\det(A)$.

(*b*) *If $B$ is the matrix that results when two rows or two columns of $A$ are interchanged, then* $\det(B) = -\det(A)$.

(*c*) *If $B$ is the matrix that results when a multiple of one row of $A$ is added to another or when a multiple of one column is added to another, then* $\det(B) = \det(A)$.

**THEOREM 2.2.4** *Let $E$ be an $n \times n$ elementary matrix.*

(*a*) *If $E$ results from multiplying a row of $I_n$ by a nonzero number $k$, then* $\det(E) = k$.

(*b*) *If $E$ results from interchanging two rows of $I_n$, then* $\det(E) = -1$.

(*c*) *If $E$ results from adding a multiple of one row of $I_n$ to another, then* $\det(E) = 1$.

**THEOREM 2.2.5** *If A is a square matrix with two proportional rows or two proportional columns, then* $\det(A) = 0$.

**THEOREM 2.3.1** *Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then*

$$\det(C) = \det(A) + \det(B)$$

*The same result holds for columns.*

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

**LEMMA 2.3.2** *If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*

$$\det(EB) = \det(E)\det(B)$$

**THEOREM 2.3.3** *A square matrix A is invertible if and only if* $\det(A) \neq 0$.

**THEOREM 2.3.4** *If A and B are square matrices of the same size, then*

$$\det(AB) = \det(A)\det(B)$$

**THEOREM 2.3.5** *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

**DEFINITION 1** If $A$ is any $n \times n$ matrix and $C_{ij}$ is the cofactor of $a_{ij}$, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by $\operatorname{adj}(A)$.

**THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint**

*If $A$ is an invertible matrix, then*

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) \tag{6}$$

It follows from Theorems 2.3.5 and 2.1.2 that if $A$ is an invertible triangular matrix, then

$$\det(A^{-1}) = \frac{1}{a_{11}}\frac{1}{a_{22}}\cdots\frac{1}{a_{nn}}$$

Moreover, by using the adjoint formula it is possible to show that

$$\frac{1}{a_{11}}, \quad \frac{1}{a_{22}}, \ldots, \quad \frac{1}{a_{nn}}$$

are actually the successive diagonal entries of $A^{-1}$ (compare $A$ and $A^{-1}$ in Example 3 of Section 1.7).

**THEOREM 2.3.7 Cramer's Rule**

*If $A\mathbf{x} = \mathbf{b}$ is a system of $n$ linear equations in $n$ unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

*where $A_j$ is the matrix obtained by replacing the entries in the $j$th column of $A$ by the entries in the matrix*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Multiplying the matrices out gives

$$\mathbf{x} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

The entry in the $j$th row of $\mathbf{x}$ is therefore

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)}$$

**THEOREM 2.3.8 Equivalent Statements**

*If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is $I_n$.*

(*d*) *A can be expressed as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every $n \times 1$ matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every $n \times 1$ matrix* $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

***Proof of Theorem 1.7.1(c)*** Let $A = [a_{ij}]$ be a triangular matrix, so that its diagonal entries are

$$a_{11}, a_{22}, \ldots, a_{nn}$$

From Theorem 2.1.2, the matrix $A$ is invertible if and only if

$$\det(A) = a_{11}a_{22} \cdots a_{nn}$$

is nonzero, which is true if and only if the diagonal entries are all nonzero.