THEOREM 1.2.1 Free Variable Theorem for Homogeneous Systems

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

THEOREM 1.2.2 *A homogeneous linear system with more unknowns than equations has infinitely many solutions.*

THEOREM 1.3.1 *If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.*

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **[Commutative law for matrix addition]**

(b) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(c) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(d) $A(B + C) = AB + AC$ **[Left distributive law]**

(e) $(B + C)A = BA + CA$ **[Right distributive law]**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(*a*) $A + 0 = 0 + A = A$

(*b*) $A - 0 = A$

(*c*) $A - A = A + (-A) = 0$

(*d*) $0A = 0$

(*e*) *If* $cA = 0$, *then* $c = 0$ *or* $A = 0$.

THEOREM 1.4.3 *If R is the reduced row echelon form of an* $n \times n$ *matrix A, then either R has a row of zeros or R is the identity matrix* I_n.

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

THEOREM 1.4.4 *If B and C are both inverses of the matrix A, then* $B = C$.

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(*a*) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(*b*) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(*c*) kA *is invertible for any nonzero scalar* k, *and* $(kA)^{-1} = k^{-1}A^{-1}$.

THEOREM 1.4.8 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(*a*) $(A^T)^T = A$

(*b*) $(A + B)^T = A^T + B^T$

(*c*) $(A - B)^T = A^T - B^T$

(*d*) $(kA)^T = kA^T$

(*e*) $(AB)^T = B^T A^T$

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

THEOREM 1.5.1 Row Operations by Matrix Multiplication

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

THEOREM 1.5.2 *Every elementary matrix is invertible, and the inverse is also an elementary matrix.*

THEOREM 1.5.3 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A is expressible as a product of elementary matrices.*

THEOREM 1.6.1 *A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

(*b*) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(*c*) kA *is invertible for any nonzero scalar* k, *and* $(kA)^{-1} = k^{-1}A^{-1}$.

THEOREM 1.6.2 *If A is an invertible* $n \times n$ *matrix, then for each* $n \times 1$ *matrix* $\mathbf{b}$, *the system of equations* $A\mathbf{x} = \mathbf{b}$ *has exactly one solution, namely,* $\mathbf{x} = A^{-1}\mathbf{b}$.

THEOREM 1.6.3 *Let A be a square matrix.*

(*a*) *If B is a square matrix satisfying* $BA = I$, *then* $B = A^{-1}$.

(*b*) *If B is a square matrix satisfying* $AB = I$, *then* $B = A^{-1}$.

THEOREM 1.6.4 Equivalent Statements

If A is an $n \times n$ *matrix, then the following are equivalent.*

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is* I_n.

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

THEOREM 1.6.5 *Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.*

THEOREM 1.7.1

(*a*) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*

(*b*) *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*

(*c*) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

(*d*) *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

THEOREM 1.7.2 *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

(*a*) *A^T is symmetric.*

(*b*) *$A+B$ and $A-B$ are symmetric.*

(*c*) *kA is symmetric.*

THEOREM 1.7.3 *The product of two symmetric matrices is symmetric if and only if the matrices commute.*

THEOREM 1.7.4 *If A is an invertible symmetric matrix, then A^{-1} is symmetric.*

THEOREM 1.7.5 *If A is an invertible matrix, then AA^T and A^TA are also invertible.*