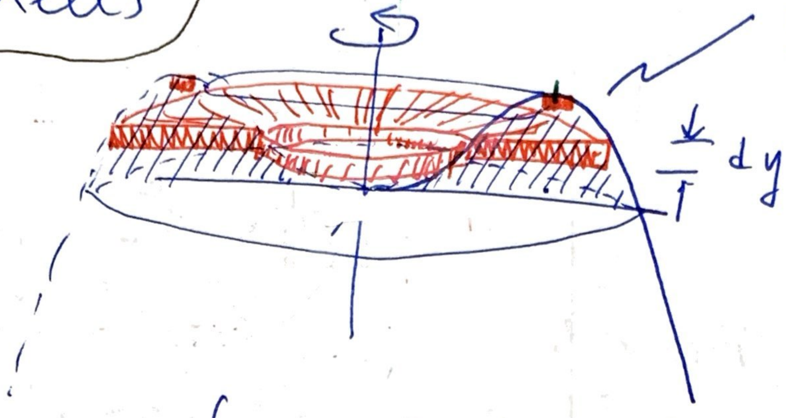
5.3 Volume by Cylindrical Shells

An alternative to washers is to use cylindrical shells

EX



$y = 2x^2 - x^3$

↳ $x = f(y)$

$x = f(y)$ is not a function

- $x = f_{RHS}$ outer
- $x = f_{LHS}$ inner

$$V = \int \underbrace{\left[\pi r_o^2 - \pi r_i^2\right]}_{f(y)} dy$$

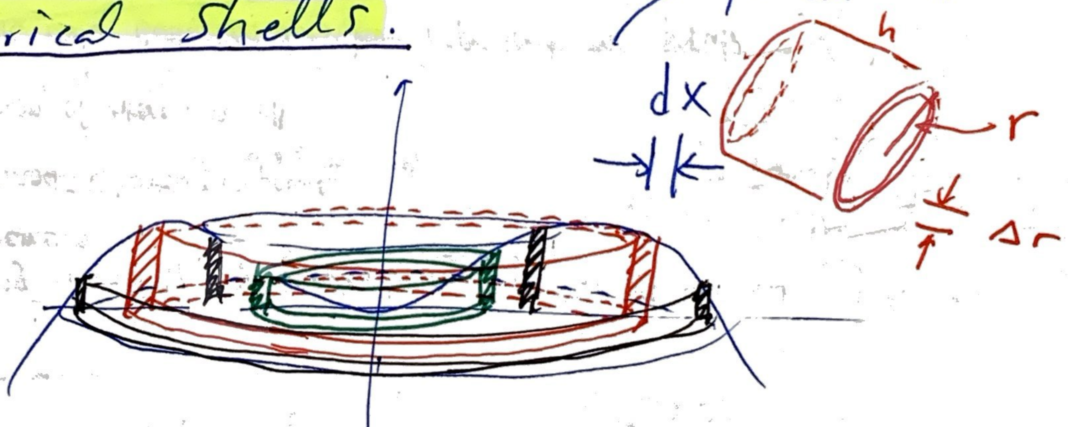
This is a difficult problem...

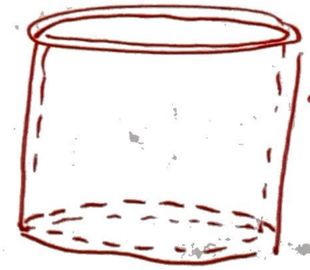
For when the geometry becomes difficult with inner and outer not being a function

TRY USING

* **Cylindrical Shells.**

$y = 2x^2 - x^3$

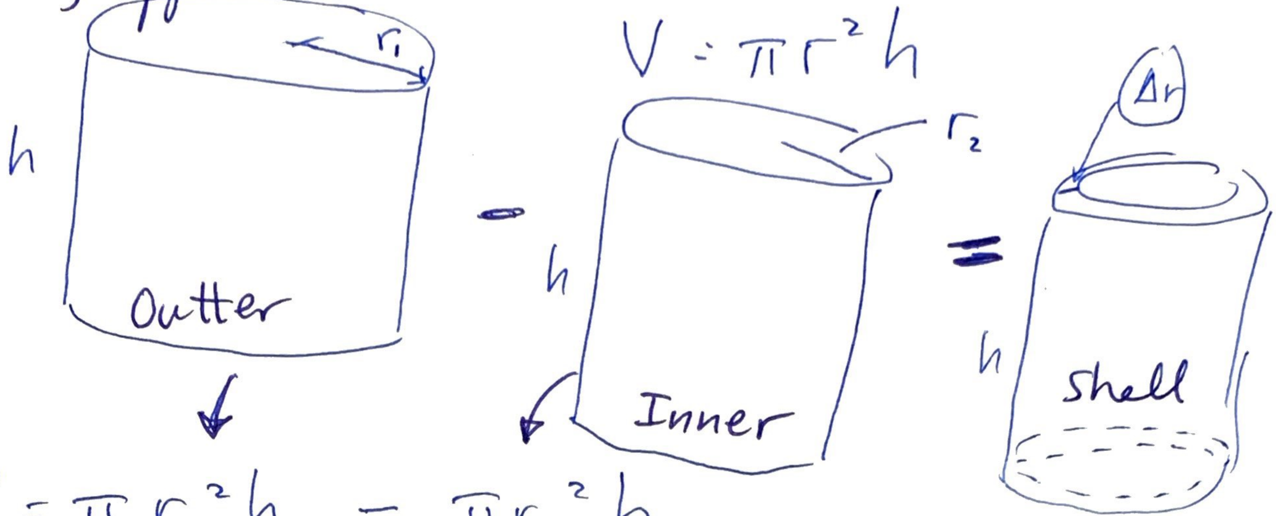




$\Delta V = ?$

⊛ Volume of a cylindrical shell

- geometry approach:



- $\Delta V = \pi r_1^2 h - \pi r_2^2 h$

$$= \pi \left[\underbrace{r_2 + \Delta r}_{r_1}\right]^2 h - \pi r_2^2 h \qquad \boxed{r_1 = r_2 + \Delta r}$$

Foil

$$= \pi \left(r_2^2 + 2 r_2 \Delta r + \Delta r^2\right) h - \pi r_2^2 h$$ cancel

$$= \cancel{\pi r_2^2 h} + \underline{\underline{2\pi r_2 \Delta r h}} + \pi \Delta r^2 h - \cancel{\pi r_2^2 h}$$

$$\boxed{\Delta V = 2\pi h r_2 \Delta r + \pi h \Delta r^2}$$

volume of shell of inner radius r_2 & thickness Δr

- let $r_2 \to r_1$: $\Delta V \to dV$

$$dV = 2\pi h r dr + \cancel{\pi h (dr)^2}$$

order of magnitude smaller than $r dr$

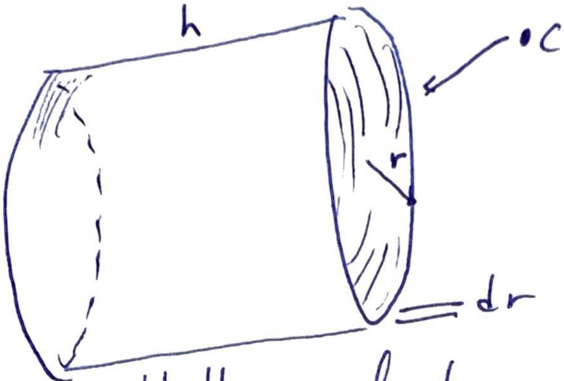
$$\boxed{dV = 2\pi h r dr}$$

if $r = x$ if $r = y$

$$V = \boxed{2\pi \int h(x)\, r(x)\, dx} = \boxed{2\pi \int h(y)\, r(y)\, dy}$$

vertical   horizontal

- physics approach :



Hollow cylinder

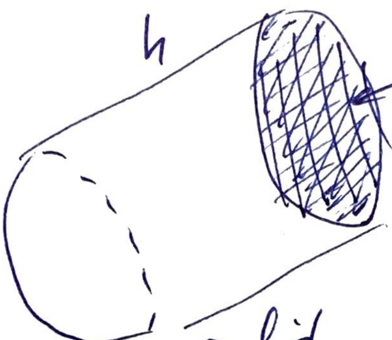
- circumference $= 2\pi r$
- Surface Area

$$S = 2\pi r h$$

- $V = S \cdot dr$

$$\boxed{V = 2\pi r h\, dr}$$

- differential approach



solid

$$V = \pi r^2 \cdot h$$

$$\frac{dV}{dr} = \pi 2 \cdot r h$$

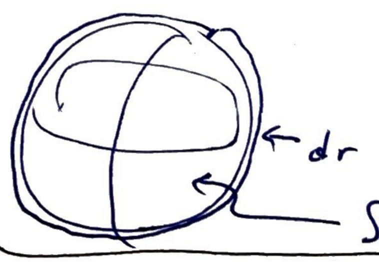
$$\boxed{dV = 2\pi h r\, dr}$$

Ex

sphere : $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = \frac{4}{3} \cdot \pi \cdot 3r^2$$

$$dV = \underbrace{4\pi r^2}_{S}\, dr \quad \text{thickness}$$



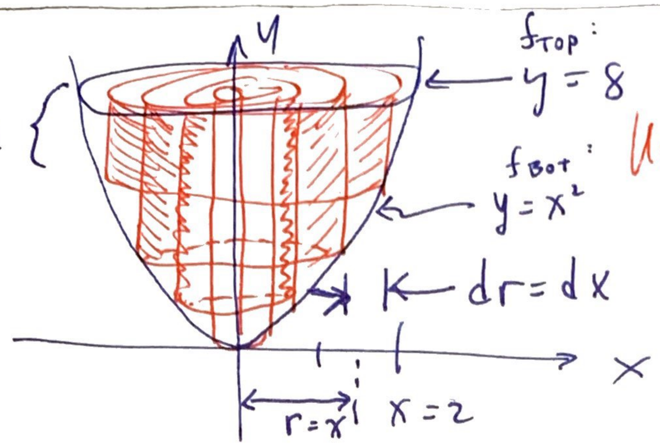
Vol of shell $dV = S \cdot dr$

EX

3

Find the volume of $y=x^2$ rotated about the y axis.

Use concentric shells.



range of x: $x=0$ to $x=2$

Details...

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \text{circ.} * \text{hieght} * \text{thickness}$$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r \cdot h \cdot \Delta r} \quad *$$

insert details

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x \cdot (f_{TOP} - f_{BOT}) \cdot \Delta x \quad *$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x \left[8 - x^2 \right] \Delta x$$

$$\boxed{V = \int_{x=0}^{x=2\sqrt{2}} 2\pi x (8-x^2)\, dx} \quad *$$

$$= 2\pi \left[\int_0^2 8x\, dx - \int_0^2 x^3\, dx \right]$$

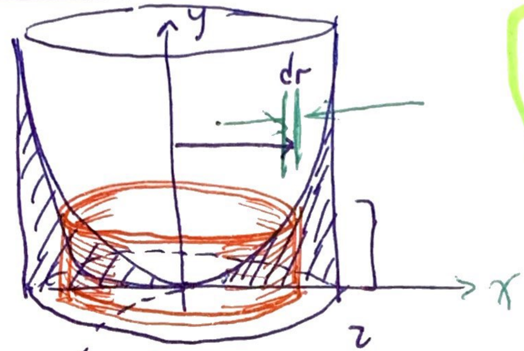
$$= 2\pi \left[\frac{8x^2}{2}\Big|_0^2 - \frac{x^4}{4}\Big|_0^2 \right]$$

$$= 2\pi \left[4(2^2-0^2) - \frac{1}{4}(2^4-0^4) \right]$$

$$= 2\pi \left[4\cdot 4 - \frac{1}{4}\cdot 16 \right]$$

$$= 2\pi \left[16 - 4 \right] = 2\pi \cdot 12 = \boxed{24\pi} \text{ cubic units}$$

EX Find the volume generated by rotating the region bound by $y = x^3$, $y = 0$ & $x = 2$ about the y-axis



↑ (i) orient cylinder

(ii) geometry step

- Thickness

  $dr = dx$ ← determines integration variable

- Radius

  $r = x$

- height of cylinder

  $h = f_{TOP}(x) - f_{BOT}(x)$

  $= x^3 - 0$

  $h = x^2$

(iii) form the integral

$$\boxed{V = \int_{r=a}^{r=b} 2\pi r h \, dr}$$

$$V = 2\pi \int_{x=0}^{2} x\left(x^3\right) dx = 2\pi \int_0^2 x^4 \, dx$$

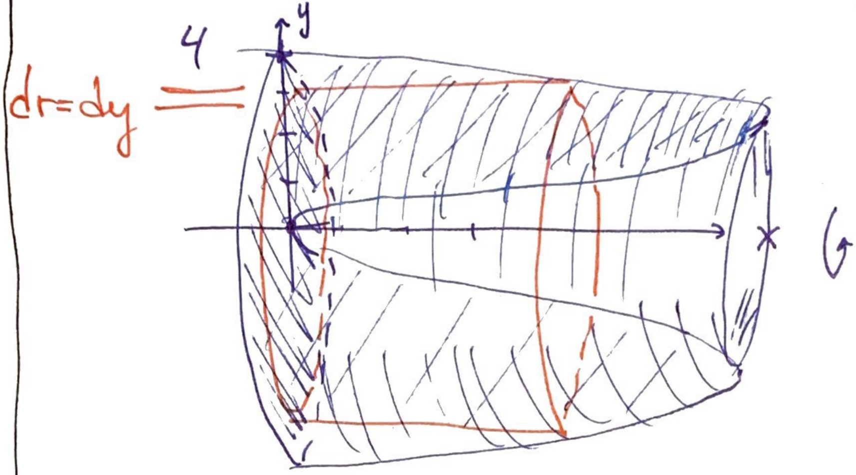
(iv) evaluate

$$V = 2\pi \left. \frac{x^5}{5} \right|_0^2$$

$$= 2\pi \left(\frac{2^5}{5} - 0 \right)$$

$$\boxed{V = \frac{64}{5}\pi} \text{ cubic units}$$

Ex Rotate about the x-axis the region bounded by $x = 4y^2 - y^3$ and $x = 0$



(i) cylinder orientation

$@x=0: \; 0 = 4y^2 - y^3$

$= y^2(4-y)$

(ii) geometry

- $dr = dy$
- $r = y$
- $h = f_r(y) - f_\ell(y)$

$= (4y^2 - y^3) - 0$

$= 4y^2 - y^3$

(iii) form integral

$$V = \int 2\pi r h \, dr$$

$$\boxed{V = 2\pi \int_{y=0}^{y=4} y(4y^2 - y^3 - 0)\,dy}$$

Set up complete

$$= 2\pi \int_0^4 (4y^3 - y^4)\,dy$$

$$= 2\pi \; \frac{\cancel{4}y^4}{\cancel{4}}\Big|_0^4 - \frac{y^5}{5}\Big|_0^4$$

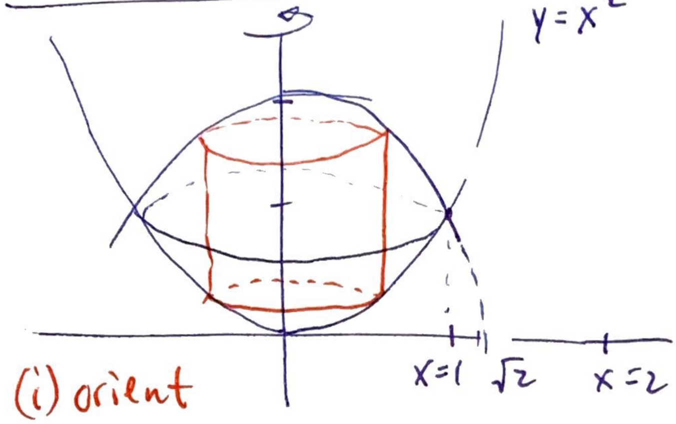
$$= 2\pi \left[256 - \frac{1024}{5}\right]$$

$$= 2\pi \left[\frac{1280 - 1024}{5}\right] = \frac{512}{5}\pi \text{ cubic units}$$

Ex

Rotate the region bounded by $\left\{ y = x^2,\ y = 2 - x^2,\ x = 1 \text{ line} \right.$ about the y-axis

6



(i) orient

(ii) geometry

- $dr = dx$
- $r = x$
- $h = f_{TOP}(x) - f_{bot}(x)$

$h = (2 - x^2) - (x^2)$

$h = 2 - 2x^2$

(iii) integral

$$V = \int 2\pi r h\, dr$$

$$V = \int_{x=0}^{x=1} 2\pi x \left(2 - 2x^2 \right) dx$$

(iv) evaluate

$$= 2\pi \int_0^1 (2x - 2x^3)\, dx$$

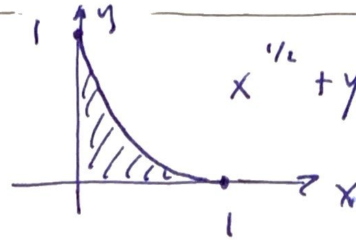
$$= 2\pi \left(\frac{\cancel{2}x^2}{\cancel{2}}\Big|_0^1 - \frac{2x^4}{4}\Big|_0^1 \right)$$

$$= 2\pi \left(1 - \frac{1}{2} \right)$$

$$= 2\pi \frac{1}{2}$$

$$= \boxed{\pi}$$

EX "astroid"



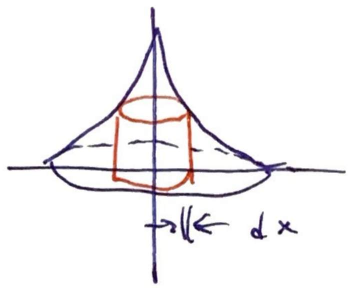
$$x^{1/2} + y^{1/2} = 1$$

$\hookrightarrow y = (1 - x^{1/2})^2$

$\hookrightarrow x = (1 - y^{1/2})^2$

* note that there is no asymptote (such required chpt 7 & 8 in calc II)

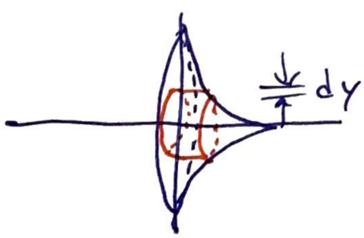
(a) rotate about y-axis



- $dr = dx$
- $r = x$
- $h = (1 - x^{1/2})^2$

$$\boxed{V = 2\pi \int_{x=0}^{x=1} x (1 - x^{1/2})^2 \, dx}$$

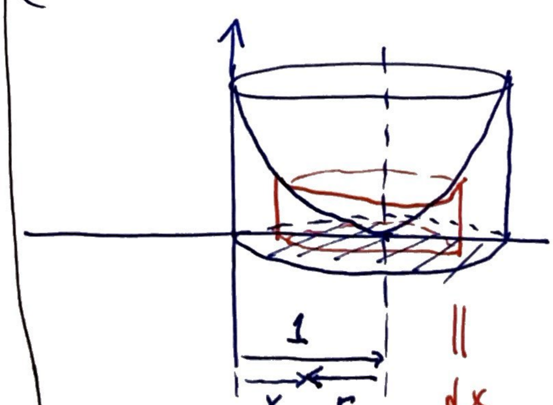
(b) rotate about x-axis



- $dr = dy$
- $r = y$
- $h = f_{right} - f_{left}$
  $= (1 - y^{1/2})^2 - 0$

$$\boxed{V = 2\pi \int_{y=0}^{y=1} y (1 - y^{1/2})^2 \, dy}$$

(c) rotate about line $x = 1$

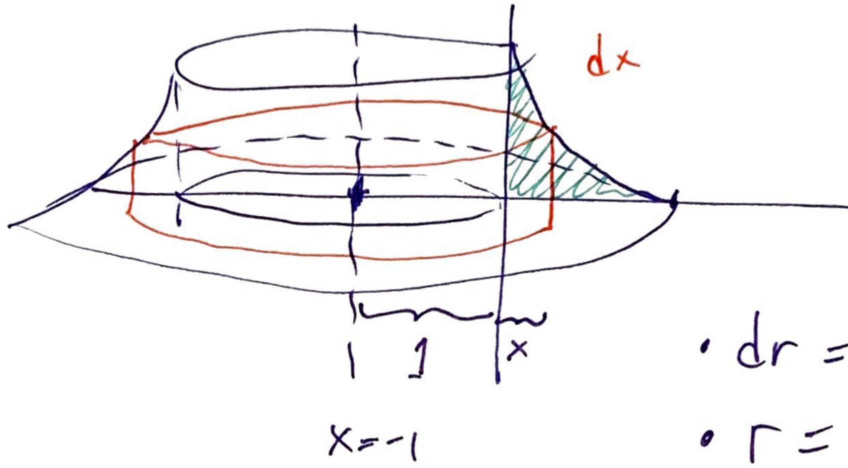


- $dr = dx$
- $r = 1 - x$
- $h = f_{Top}(x) - f_{Bot}(x)$
  $= (1 - x^{1/2})^2$

radius height thick.

$$V = 2\pi \int_{x=0}^{x=1} (1 - x)(1 - x^{1/2})^2 \, dx$$

(d) rot. about $x = -1$



- $dr = dx$
- $r = 1 + x$
- $h = (1 - x^{1/2})^2$

$$V = 2\pi \int_{x=0}^{x=1} (1+x)(1-x^{1/2})^2 \, dx$$