

4.5 The Substitution Rule {chain rule in "Reverse"} ^①

We know that $\int \cos(x) dx = \sin(x) + C$. But what is $\int \sin(\sqrt{2}x) dx = ?$

I This is where we introduce the "u-sub" rule

Ex $\int \sin(\sqrt{2}x) dx$

let $u = \sqrt{2}x$ then $du = \sqrt{2} dx$ or $dx = \frac{du}{\sqrt{2}}$

Now the integral becomes

$$\int \sin(u) \left(\frac{du}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \int \sin(u) du$$

$$= \frac{1}{\sqrt{2}} (\cos(u) + C) \quad \text{unsubstitute}$$

$$= \frac{1}{\sqrt{2}} (\cos(\sqrt{2}x) + C)$$

$$\int \sin(\sqrt{2}x) dx = \frac{1}{\sqrt{2}} \cos(\sqrt{2}x) + C$$

1. Use a u-substitution to evaluate

$$\int x^2 \sin(x^3) dx$$

always present the u-sub with its differential

$$u = x^3$$

$$du = 3x^2 dx$$

$$dx = \frac{du}{3x^2}$$

Mixed integral
Keep going

$$= \int x^2 \sin(u) \left(\frac{du}{3x^2}\right)$$

$$= \frac{1}{3} \int \sin(u) du$$

make sure after the u-sub the entire integral is in "u" before proceeding

$$= \frac{-1}{3} \cos(u) + C$$

Done? No!

$$= \frac{-1}{3} \cos(x^3) + C$$

Done!

you must put the results back into the original variable

2

EX evaluate $\int 3x^2 \sqrt{1+x^3} dx$

let $u = 1+x^3$ then $du = 3x^2 dx$

So the integral becomes

$$\int \sqrt{u} (3x^2 dx)$$

$$= \int \sqrt{u} du$$

$$= \frac{u^{\frac{1}{2}+1} = \frac{3}{2}}{\frac{1}{2}+1} + C$$

unsubstitute

$$= \frac{2(\sqrt{1+x^3})^3}{3} + C$$

Ex Evaluate $\int x^3 \sqrt{x^2+1} dx$

let $u = x^2 + 1 \rightarrow du = 2x dx \rightarrow dx = \frac{du}{2x}$

$$\int x^3 \sqrt{u} \left(\frac{du}{2x}\right)$$

$$= \frac{1}{2} \int x^2 \sqrt{u} du$$

← mixed integral. How do we convert the "x" to "u" to?

but $x^2 = u - 1$

$$= \frac{1}{2} \int (u-1) \sqrt{u} du$$

← fully a "u" integral now

$$= \frac{1}{2} \int u \sqrt{u} du - \frac{1}{2} \int \sqrt{u} du$$

$$= \frac{1}{2} \int u^{3/2} du - \frac{1}{2} \int u^{1/2} du$$

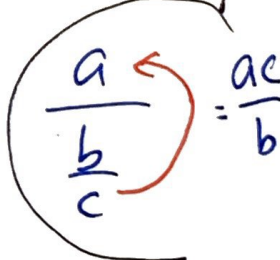
$$= \frac{1}{2} \frac{u^{3/2+1}}{3/2+1} - \frac{1}{2} \frac{u^{1/2+1}}{1/2+1} + C$$

$$= \frac{1}{2} \frac{u^{5/2}}{5/2} - \frac{1}{2} \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{2} \cdot \frac{2}{5} u^{5/2} - \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$$

u sub

$$= \frac{1}{5} (\sqrt{x^2+1})^5 - \frac{1}{3} (\sqrt{x^2+1})^3 + C$$



Theory

Thm If $u=g(x)$ is diffble on range I
 and if $f(x)$ is continuous on I also
 then $\int f(g(x))g'(x)dx = \int f(u)du$

proof:

I. $\int f(g(x))g'(x)dx$

• Note $\frac{d}{dx} [F(g(x))] = F'(g) \cdot g'$ chain Rule

• now let $F' = f$
 so then $I = \int \overbrace{f(g(x))}^{F'} \underbrace{g'(x)dx}_{dg}$

$$\frac{d}{dx} \int f = f$$

becomes $= \int F'(g) dg$

$$\frac{dg(x)}{dx} dx = dg$$

$= F(g) + C$ by FTC-I

Now let $u=g(x) \rightarrow du=g'(x)dx$

then $\int F'(g(x))g'(x)dx$

$= \int F'(u) du$

but $F' = f$

LHS

RHS

$$\int f(g(x))g'(x)dx = \int f(u)du$$

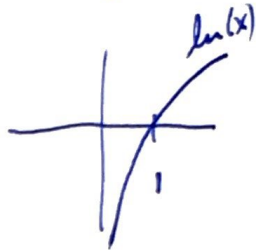
Q.E.D.

In chapter 6 we reveal that

$$\frac{d \ln(x)}{dx} = \frac{1}{x}$$

⑤

$$\begin{cases} \ln(x) = \log_e(x) \\ \log(x) = \log_{10}(x) \end{cases}$$



So in reverse (antiderivation)

$$\int \frac{1}{x} dx = \ln(x) + C$$

Ex

Find $\int \tan(x) dx$

wow, what to do here?

use def of tangent

$\int \frac{\sin(x) dx}{\cos(x)}$, now let $u = \cos(x)$ so $du = -\sin(x) dx$

$$\begin{aligned} &= \int \frac{-du}{u} \\ &= -\ln(u) + C \\ &= -\ln(\cos(x)) + C \\ &= \ln(\cos(x))^{-1} + C \\ &= \ln\left(\frac{1}{\cos(x)}\right) + C \\ &= \ln(\sec(x)) + C \end{aligned}$$

$$b \cdot \log(a) = \log(a^b)$$

$$\begin{cases} \text{let } b = -1 \\ \log(a) = \log(a^{-1}) \\ = \log\left(\frac{1}{a}\right) \end{cases}$$

So

$$\int \tan(x) dx = \ln|\sec(x)| + C$$

EX #4 on CW 11 b

$$I = \int \sec^2(4t) [3 - \tan(4t)]^3 dt$$

let $u = 4t$
 $du = 4dt$

make sure to completely convert the integral to one in "u"

$$= \int \sec^2(u) [3 - \tan(u)]^3 \left(\frac{du}{4}\right)$$

let $w = 3 - \tan(u)$
 $dw = -\sec^2(u) du$

make sure to completely convert to w

$$= \frac{1}{4} \int [3 - \tan(u)]^3 (\sec^2(u) du)$$

$$= \frac{1}{4} \int w^3 (-dw)$$

$$= -\frac{1}{4} \left(\frac{w^4}{4}\right) + C$$

unsub w

$$= -\frac{1}{16} [3 - \tan(u)]^4 + C$$

unsub u

$$= -\frac{1}{16} [3 - \tan(4t)]^4 + C$$

make sure the result are in the original variable

II Definite Integrals

For definite Integrals we have two choices

A. Evaluate the integral, w/o limits, via a u-sub, then unsubstitute and reply the limits.

B. Change the limits when we do the u-sub.

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(b)}^{u=g(a)} f(u)du$$

Ex of method A.

$$1 + \tan^2 = \sec^2$$

$$\int_0^{\pi/3} \frac{\sin(z) + \sin(z)\tan^2(z)}{\sec^2(z)} dz$$

clean up, maybe we will see what to do next...

$$= \int_0^{\pi/3} \frac{\sin(z) \overbrace{(1 + \tan^2(z))}^{\sec^2(z)}}{\sec^2(z)} dz$$

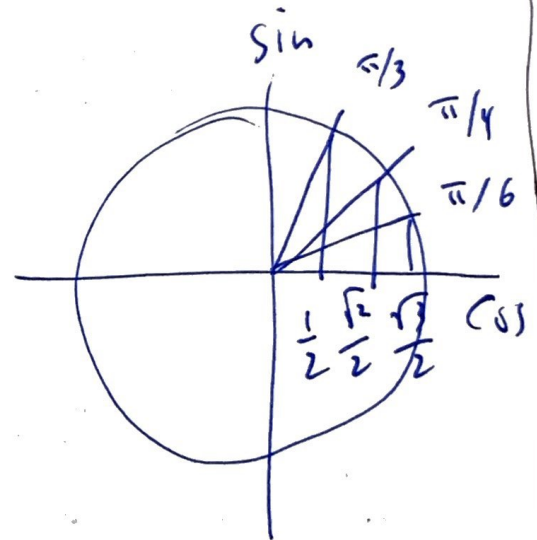
$$= \int_0^{\pi/3} \sin(z) dz$$

$$= -\cos(z) \Big|_0^{\pi/3}$$

$$= -\cos\left(\frac{\pi}{3}\right) - (-\cos(0))$$

$$= -\frac{1}{2} - (-1)$$

$$= \frac{1}{2}$$



Hum

No u-sub needed... after all...

Ex

Evaluate $\int_{x=1}^4 3x^2 \sqrt{1+x^3} dx$

Method
A

$u = 1 + x^3 \rightarrow du = 3x^2 dx$ { previous example }

$\int_{x=1}^{x=4} \sqrt{u} du$

$= \frac{u^{3/2}}{3/2} \Big|_{x=1}^{x=4}$ integrate

$= \frac{2(\sqrt{1+x^3})^3}{3} \Big|_{x=1}^{x=4}$ un-substitute

evaluate limits...

$= \frac{2}{3} \left[(\sqrt{1+4^3})^3 - (\sqrt{1+1^3})^3 \right]$

$= \frac{2}{3} \left[\sqrt{65}^3 - \sqrt{2}^3 \right]$

$(\sqrt{a})^3 = \sqrt{a} \sqrt{a} \sqrt{a}$
 $= a\sqrt{a}$

$= \frac{2}{3} 65\sqrt{65} - \frac{2}{3} 2\sqrt{2}$

$= \frac{130\sqrt{65}}{3} - \frac{4\sqrt{2}}{3}$ done

next approach ...

Ex cont.

Do it again but now use

8

Method

B

$$\int_{x=1}^{x=4} 3x^2 \sqrt{1+x^3} dx$$

$$\text{let } u = 1+x^3, \quad du = 3x^2 dx$$

$$\int_{x=1}^{x=4} \sqrt{u} du$$

$$= \int_{u=1+(1)^3}^{u=1+(4)^3} \sqrt{u} du$$

$$= \int_{u=2}^{u=65} u^{1/2} du$$

Totally converted to "u"

No more "x"

$$= \int_2^{65} u^{1/2} du$$

$$= \frac{u^{3/2}}{3/2} \Big|_2^{65}$$

$$= \frac{2}{3} \left[65^{3/2} - 2^{3/2} \right]$$

$$= \frac{2 \cdot 65\sqrt{65}}{3} - \frac{2 \cdot 2\sqrt{2}}{3}$$

but now we convert the limits to "u" also!

III

Symmetry and Integrals

If $f(x)$ is continuous on a symmetric interval $[-a, a]$ then

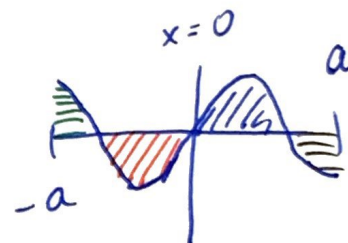
(a) If $f(x) = \text{even function}$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$



(b) If $f(x) = \text{odd function}$

$$\int_{-a}^a f(x) dx = 0$$



proof for (b): Recall $\int_{-a}^0 f(x) dx = - \int_0^{-a} f(x) dx$

Then $\int_{-a}^a f dx = \int_{-a}^0 f dx + \int_0^a f dx$

$$= - \int_{x=0}^{x=-a} f(x) dx + \int_0^a f dx$$

let $u = -x$
 $du = -dx$

$$= - \int_{u=a}^{u=0} f(-u) (-du) + \int_0^a f(x) dx$$

$$= \int_{u=0}^a -f(u) du + \int_{x=0}^a f(x) dx = -I + I = \boxed{0}$$

EX

Evaluate

10

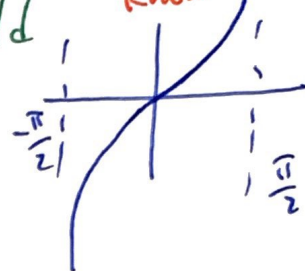
$$\int_{-\pi/4}^{\pi/4} (x^3 + x^4 \tan(x)) dx$$

no method will work that we know

$$= \int_{-\pi/4}^{\pi/4} x^3 dx + \int_{-\pi/4}^{\pi/4} x^4 \tan(x) dx$$

← even
← odd

↑ odd
↑ odd



Sym interval

$$= 0 + 0$$

Even · odd = odd
 odd · odd = even

EX

$$\int_0^1 x \sqrt{1+x^2} dx$$

Just another u-sub example

$$u = 1+x^2, du = 2x dx$$

$$= \int_{u=1+0^2}^{u=1+1^2} \sqrt{u} \left(\frac{du}{2}\right)$$

$$= \frac{1}{2} \int_1^2 \sqrt{u} du$$

$$= \frac{1}{2} \frac{u^{3/2}}{3/2} \Big|_1^2$$

$$= \frac{1}{3} [\sqrt{2}^3 - \sqrt{1}^3]$$

$$= \frac{2\sqrt{2}}{3} - \frac{1}{3}$$

