$\boxed{4.4}$ Indefinite Integrals

Recall $\int_a^b f(x)dx$ is a number. This is called Definite Integral

Def: $\int f(x)dx$ is a function of x. We call this type of integral an Indefinite Integral

$\boxed{I}$ Some indefinite integrals

$$\boxed{\int f(x)dx = F(x) + C}$$

where $\dfrac{dF(x)}{dx} = f(x)$

The integral of a function is just its anti-derivation

- Recall we decifered that $\boxed{\int x^n dx = \dfrac{x^{n+1}}{n+1} + C}$
- verify: $\dfrac{d\left[\dfrac{x^{n+1}}{n+1} + C\right]}{dx} = \dfrac{1}{n+1}\dfrac{dx^{n+1}}{dx} + \cancel{\dfrac{dC}{dx}} \nearrow 0$

$$= \frac{1}{\cancel{n+1}}\cancel{(n+1)}x^{(n+1)-1} = \boxed{x^n}$$

- Knowing that the integrand is derivative of the results, we can create quick formulas that we look-up {memorize}
- Our Table of Integrals (so far)

$$\boxed{\int k\,dx = kx + c} \qquad \text{since } \frac{d(kx+c)}{dx} = k$$

$$\boxed{\int x\,dx = \frac{x^2}{2} + c} \qquad \text{since } \frac{d\,x^2/2 + c}{dx} = x$$

$$\boxed{\int x^2\,dx = \frac{x^3}{3} + c}$$

$$\vdots$$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c}$$

$$\boxed{\int \sin(x)\,dx = -\cos(x) + c} \qquad \text{since } \frac{d - \cos(x)}{dx} = \sin(x)$$

$$\boxed{\int \cos(x)\,dx = \sin(x) + c}$$

$$\boxed{\int \tan(x)\,dx = \ln|\sec(x)| + c} \leftarrow \text{chpt 6}$$

$$= -\ln|\cos(x)| + c$$

$$\boxed{\int \sec^2(x)\,dx = \tan(x) + c}$$

since $\frac{d\tan}{dx} = \sec^2$

$$\boxed{\int \csc^2(x)\,dx = -\cot(x) + c}$$

since $\frac{d[-\cot(x)+c]}{dx} = [\csc(x)]^2$

$$\boxed{\int \sec(x)\tan(x)\,dx = \sec(x) + c}$$

b/c $\frac{d\sec(x)}{dx} = \frac{d\ 1/\cos(x)}{dx} = \frac{d[\cos(x)]^{-1}}{dx} = -1[\cos(x)]^{-2}\cdot\frac{d\cos(x)}{dx}$

$= -\frac{1}{\cos^2(x)}(-\sin(x)) = \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)}$

$= \underline{\underline{\tan(x)\sec(x)}}$

$$\boxed{\int \csc(x)\cot(x) = -\csc(x) + c}$$

Ex Integrate $\int \frac{x^3-2\sqrt{x}}{x}\,dx$

$$= \int\left(\frac{x^3}{x} - 2\frac{\sqrt{x}}{x}\right)dx$$

$\frac{1}{x}(x^3-2\sqrt{x})$

$$= \int\left(x^2 - 2x^{1/2-1}\right)dx$$

$\boxed{\frac{a^n}{a^m} = a^{n-m}}$

$$= \int\left(x^2 - 2x^{-1/2}\right)dx$$

$$= \int x^2\,dx - 2\int x^{-1/2}\,dx$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$$= \frac{x^{2+1}}{2+1} - 2\left[\frac{x^{-1/2+1}}{-1/2+1}\right] + c$$

$$= \frac{x^3}{3} - 2\left[\frac{x^{1/2}}{1/2}\right] + c$$

$\frac{1}{\frac{1}{2}} = 2$

$$= \frac{x^3}{3} - x^{1/2} + c$$

$$= \boxed{\frac{1}{3}x^3 - \sqrt{x} + c}$$

Ex Integrate $\int \frac{\sin(x)}{\cos^2(x)}\,dx$

Hint: convert to sec, tan

$$= \int \sin(x)\sec^2(x)\,dx$$

OR

$$= \int \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)}\,dx$$

$$= \int \tan(x)\sec(x)\,dx$$

$$= \sec(x) + c$$

So $\boxed{\int \frac{\sin(x)}{\cos^2(x)}\,dx = \sec(x) + c}$

Ex Evaluate $\int \frac{\sin(2x)}{\sin(x)}\,dx$

Hint: use a trig identity

$\sin(2x) = 2\sin(x)\cos(x)$

$$= \int \frac{2\cancel{\sin(x)}\cos(x)}{\cancel{\sin(x)}}\,dx$$

$$= 2\int \cos(x)\,dx$$

$$= \boxed{2\sin(x) + c}$$

So $\int \frac{\sin(2x)}{\sin(x)}\,dx = \underline{\underline{2\sin(x) + c}}$

Ex Integrate

$$\int \sec(t)\left[\sec(t)+\tan(t)\right]dt$$

$$= \int \sec^2(t)\,dt + \int \sec(t)\tan(t)\,dt$$

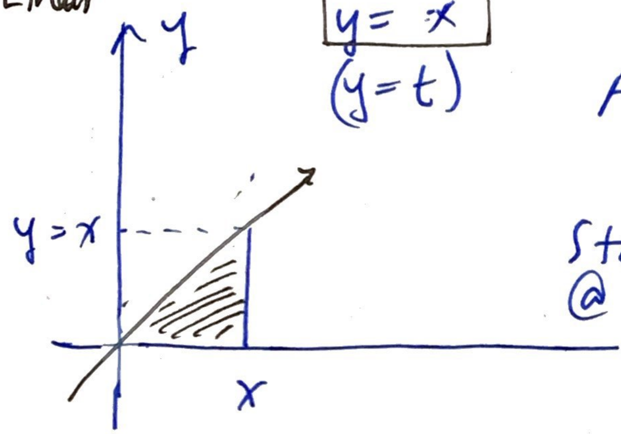
$$= \boxed{\tan(t) + \sec(t) + c}$$

Area

Consider area under the line $y = x$

Q: How does area grow as we move x further out on the x-axis?

Ex Linear



$\boxed{y = x}$

$(y = t)$

$$\text{Area} = \int_a^b f(t)\,dt$$

Start @ $x = 0$

$$= \int_0^x (t)\,dt$$

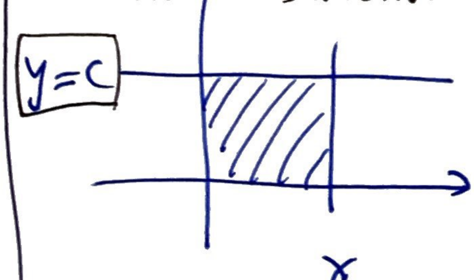
$$= \left.\frac{t^2}{2}\right|_0^x$$

$$\boxed{A(x) = \frac{x^2}{2}}$$

$$A_\Delta = \frac{1}{2} b \cdot h$$

Ex If $y = c$ what is $A(x)$?

Constant function



$$A(x) = \int_0^x c\,dt$$

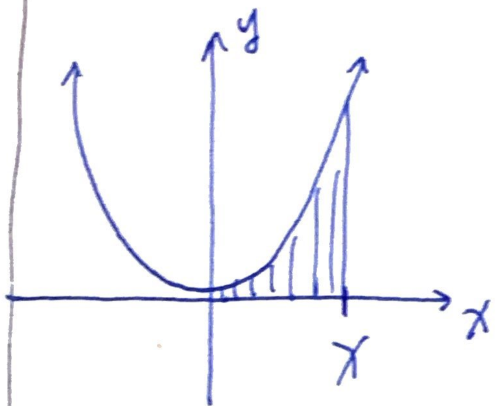
$$= c\,t\Big|_0^x$$

$$= c(x - 0)$$

$$\boxed{A(x) = cx}$$

$$A = W \cdot H$$

Ex If $y = x^2$ what is $A(x)$?



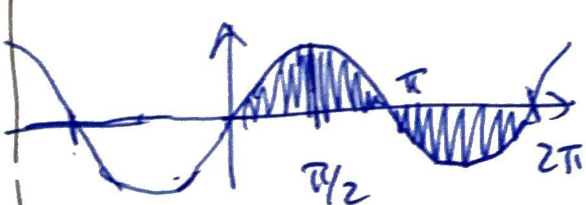
$$A(x) = \int_0^x t^2\,dt$$

$$= \frac{t^3}{3}\Big|_0^x$$

$$= \frac{x^3}{3} - \frac{0^3}{3}$$

$$\boxed{A(x) = \frac{x^3}{3}}$$

Ex Find the accumulation of $f(x) = \sin(x)$ from 0 to 2π:

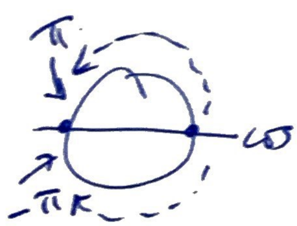


$$f = \frac{dF}{dx}$$

$$\boxed{I = \int_0^{2\pi} \sin(x)\,dx}$$

$F(b)$ $F(a)$

$$= [-\cos(x)]\Big|_{2\pi} - [-\cos(x)]\Big|_0$$



$$= -\cos(2\pi) + \cos(0)$$

$$= -1 + 1 = \boxed{0}\ !!!$$

Net accumulation is zero

* We will see later that $A \neq \int_a^b f(x)\,dx$ when $f(x) < 0$. We need absolute values for when $f(x) < 0$ to get area between f & x-axis.

II Net Change

- FTC · II:

$$\boxed{\int_a^b F'(t)\,dt = F(b)-F(a)}$$

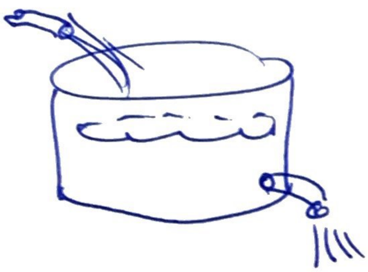
rate of change

Net change

- lets write $f(x)$ as $F'(x)$

⊛ Volume of Liquid in a tank.

- let $V(t)$ = volume of liquid in a tank at time "t"



Combined rate of change of the volume is $\dfrac{dV(t)}{dt}$

Then the volume change at a further time follows this formula (FTC II)

$$\boxed{V(t_2) - V(t_1) = \int_{t_1}^{t_2} V'(t)\,dt}$$

move to other side

(+) means tank will fill

EX let $V'(t) = \dfrac{1}{t^2}$ if $t > 1$ min. Find the volume in the tank at $t = 3$ min, start@ $t_1 = 1$ min

$$V(3) = V(1) + \int_1^3 \left(\frac{1}{t^2}\right) dt$$

$$V(3) = \frac{1}{t^2}\Big|_{t=1} + \int_1^3 t^{-2}\,dt$$

$$V(3) = \frac{1}{1^2} + \frac{t^{-2+1}}{-2+1}\Big|_1^2$$

$$= 1 + \left(\frac{t^{-1}}{-1}\right)\Big|_1^2$$

$$= 1 - \frac{1}{t}\Big|_1^2$$

$$= 1 - \left(\frac{1}{2} - \frac{1}{1}\right)$$

$$= 2 - \tfrac{1}{2} = \boxed{3/2 \text{ gal}}$$

Instantaneous density (10)

⊛ Density

- Linear Density = mass / length $\Rightarrow$ $\boxed{\rho = \dfrac{dm(x)}{dx}}$

- Variable density $\rho = f(x)$

(ex)

50% lead
50% Al

80% lead
20% Al

mass of a piece of rod:

$\Delta m = \rho(x) \cdot \Delta x$

So $\Delta m = \left(\dfrac{dm(x)}{dx}\right) \Delta x$

- Total mass between a & b $= \displaystyle\sum_{i=1}^{n} \rho(x_i) \cdot \Delta x$

- net mass @ $x = b$ is

integrate

$$\boxed{m(b) = m(a) + \int_a^b \rho(x)\,dx}$$

ex If a rod's density falls of as $\rho(x) = \dfrac{\rho_0}{\sqrt{x}}$

Find the mass between $a = 1\,\text{cm}$ & $b = 8\,\text{cm}$

$$\Delta m = \int_a^b \rho(x)\,dx$$

$$\Delta m = \int_1^8 \frac{\rho_0}{\sqrt{x}}\,dx$$

$$= \rho_0 \int_1^8 x^{-1/3}\,dx$$

$$\Delta m = \rho_0 \left.\frac{x^{-1/3+1}}{-1/3+1}\right|_{x=1}^{x=8}$$

$$\Delta m = \left.\frac{3\rho_0 x^{2/3}}{2}\right|_1^8$$

$$\Delta m = \frac{3}{2}\rho_0\left(\sqrt[3]{8}^{\,2} - \sqrt[3]{1}^{\,2}\right)$$

$$\Delta m = \frac{3}{2}\rho_0\left[4-1\right] = \boxed{\frac{9}{2}\rho_0}$$

⊛ Electrical Charge

- Current in a wire, or capacitor, is the amount of charge flowing across some reference point per unit time

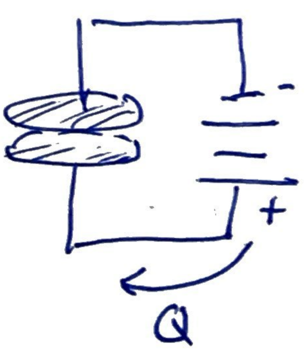
$$I \equiv \frac{\Delta Q}{\Delta t}$$

Instantaneous current is $\boxed{I = \frac{dQ}{dt}}$

- What does $\displaystyle\int_{t=a}^{t=b} I(t)\,dt$ mean?

Ans: $\boxed{\displaystyle\int_a^b \left(\frac{dQ}{dt}\right) dt = Q(b) - Q(a)}$ via the FTC·II

$\boxed{\text{EX}}$ A capacitor accumulates charge, and while doing so appears to have a current flowing across its plates.



Q: What is the net change of charge if $I(t) = \frac{I_0}{t^3}$ from $t = 1_s$ to $t = 3_s$

$$\Delta Q = \int_{t=1}^{3} \left(\frac{I_0}{t^3}\right) dt$$

$$= I_0 \int_1^3 t^{-3}$$

$$= I_0 \left.\frac{t^{-2}}{-2}\right|_1^3$$

$$= \frac{-I_0}{2}\left(\frac{1}{3^2} - \frac{1}{1^2}\right)$$

$$= \frac{-I_0}{2}\left(\frac{1}{9} - 1\right) = \frac{I_0}{\cancel{2}} \cdot \frac{\cancel{8}^{\,4}}{9}$$

$$\boxed{\frac{I_0 4}{9}}$$

Economics

- If $C(x)$ is the cost to produce x widgets of a commodity, like cell phones, say, then the marginal cost is $C'(x)$ and represents the cost per unit after x widgets have been produced. {A.K.A. Instantaneous Cost/unit @ x}
- The net change of Cost from X_1 widgets to X_2 widgets is then

$$\boxed{C(X_2) - C(X_1) = \int_{X_1}^{X_2} C'(x)\,dx}$$

Ex let $4 - 0.02x$ be the marginal cost to produce x toothbrushes. What is the cost expended from the 10th brush to the 30th brush

$C'(x) = 4 - 0.02x$

$$\Delta C = \int_{10}^{30} (4 - 0.02x)\,dx$$

$$= 4x\Big|_{10}^{30} - 0.02\frac{x^2}{2}\Big|_{10}^{30}$$

$$= 4(30-10) - \frac{0.02}{2}(30^2 - 10^2)$$

$$= 80 - 0.01(900 - 100)$$

$$= 80 - 0.01(800)$$

$$= 80 - 8$$

$$= \boxed{72\ \$}$$