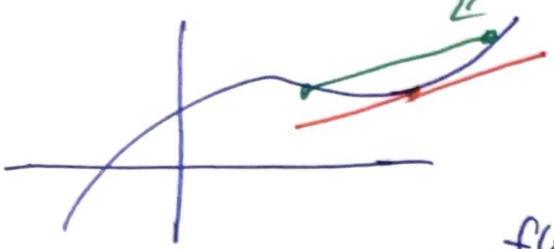
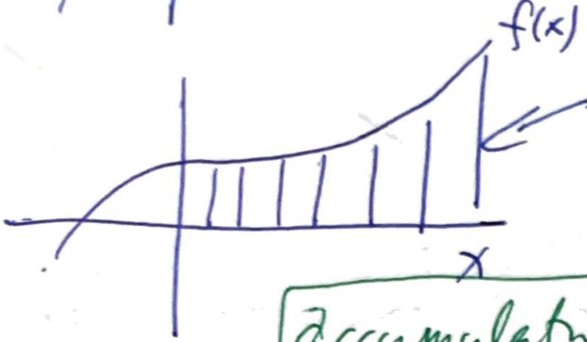
Chapter 4 Integration {accumulation}

Calculus is largely diff'n and integration

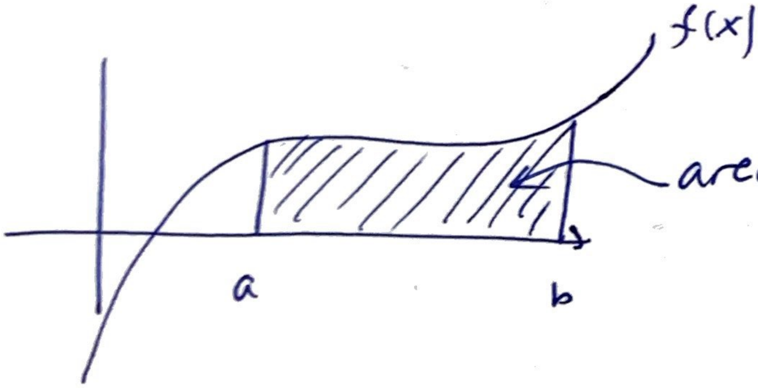
- **Diff'n**



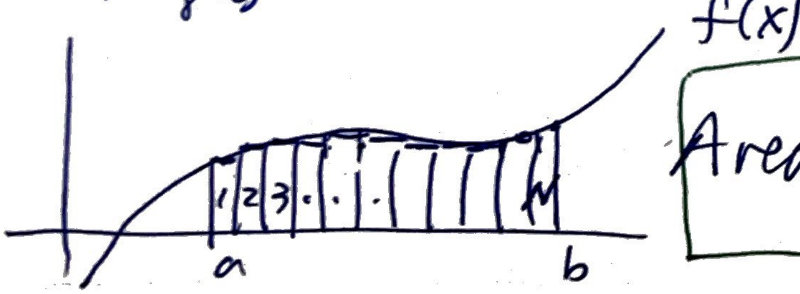
- **Integration**



4.1 Area and distance



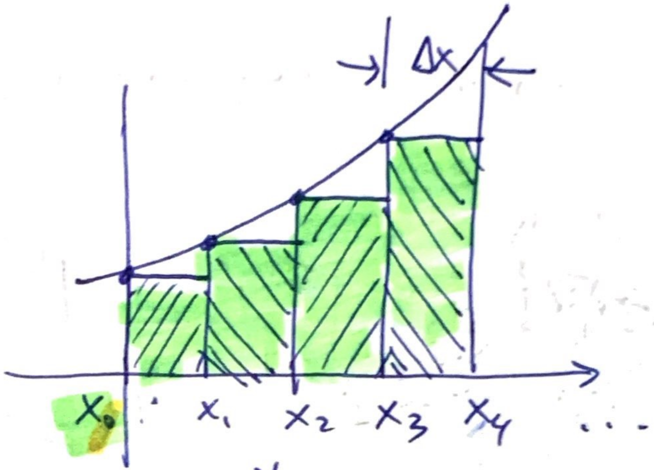
We Approx area by slicing the area into rectangles



$$\boxed{\text{Area} \approx \sum_{i=1}^{N} \text{rectangles}}$$

LHS rectangles

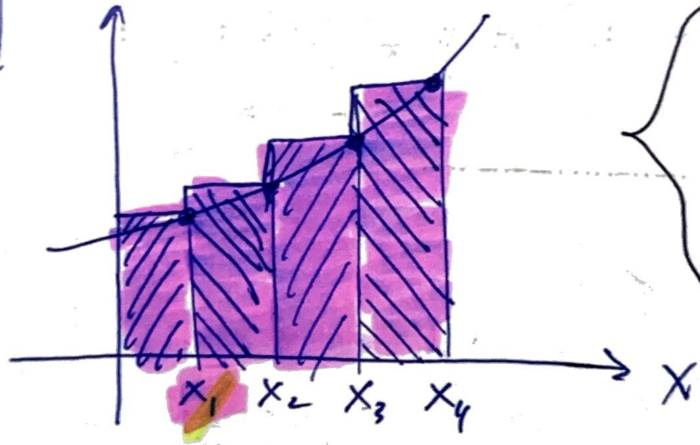
left hand side



when concave up we under-estimate area

$$\text{Area} \approx \sum_{i=0}^{N} \underbrace{f(x_i)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}$$

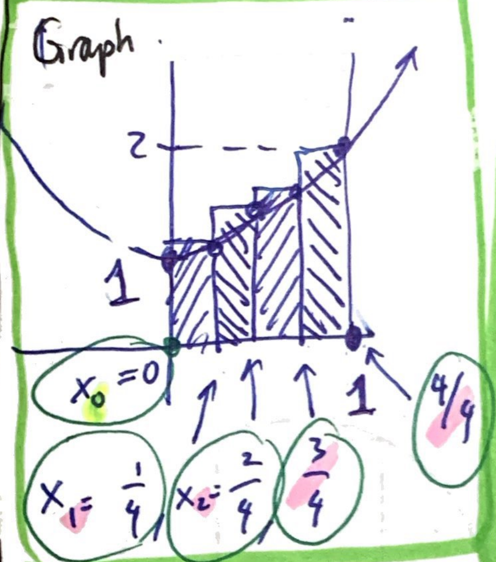
RHS rectangles



when concave up we over-estimate area

$$\text{Area} \approx \sum_{i=1}^{N} f(x_i)\,\Delta x$$

Ex Consider $y = x^2 + 1$. Divide the region $[0, 1]$ into 4 rectangles and use (a) the RHS approach to approx area then (b) the LHS approach



$$\Delta x = 1/4 = \frac{(1-0)}{4} = \frac{b-a}{N}$$

- $f(x_0) = f(0) = 0^2 + 1 = 16/16$
- $f(x_1) = f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + 1 = 17/16$
- $f(x_2) = f(2/4) = \left(\frac{2}{4}\right)^2 + 1 = 20/16$
- $f(x_3) = f\left(\frac{3}{4}\right) = (3/4)^2 + 1 = 25/16$
- $f(x_4) = f\left(\frac{4}{4}\right) = (4/4)^2 + 1 = 32/16$

(a) RHS

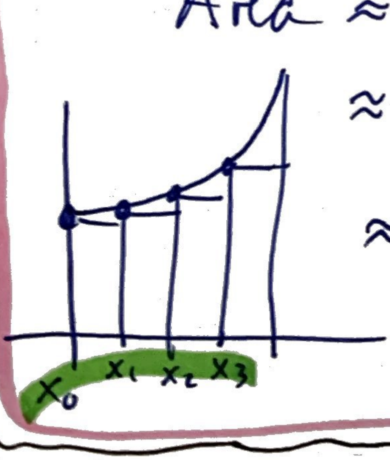
$$\text{Area} \approx f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x$$

$$\approx \left(\frac{17}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{20}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{25}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{32}{16}\right)\left(\frac{1}{4}\right)$$

$$\approx \boxed{\frac{47}{32}} \text{ square units.}$$

(b) LHS

$$\text{Area} \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$



$$\approx \left(\frac{16}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{17}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{20}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{25}{16}\right)\left(\frac{1}{4}\right)$$

$$\approx \frac{16 + 17\cdot 1 + 20\cdot 1 + 25\cdot 1}{16\cdot 4}$$

$$\approx \boxed{39/32} \text{ sq units}$$

True answer in between

$$\frac{39}{32} < \text{Area} < \frac{47}{32}$$

$\boxed{\text{Ex}}$ cont.

lets average these two $\left(\frac{47+39}{2}\right)/32 = \boxed{\frac{43}{32}}$

Q: is this a better approximation ≈ 1.34375

only one digit approx

Exact answer

Learn in 4.2

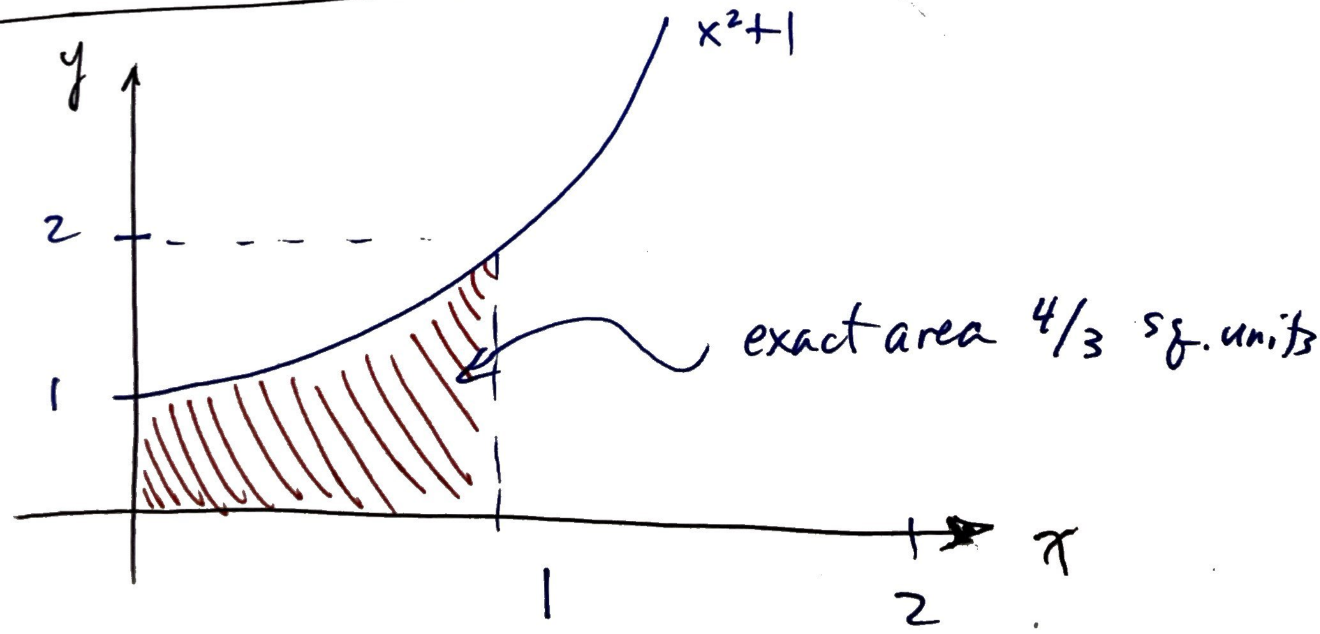
$$A = \int_0^1 f(x)\,dx$$

$$= \int_0^1 (x^2+1)\,dx$$

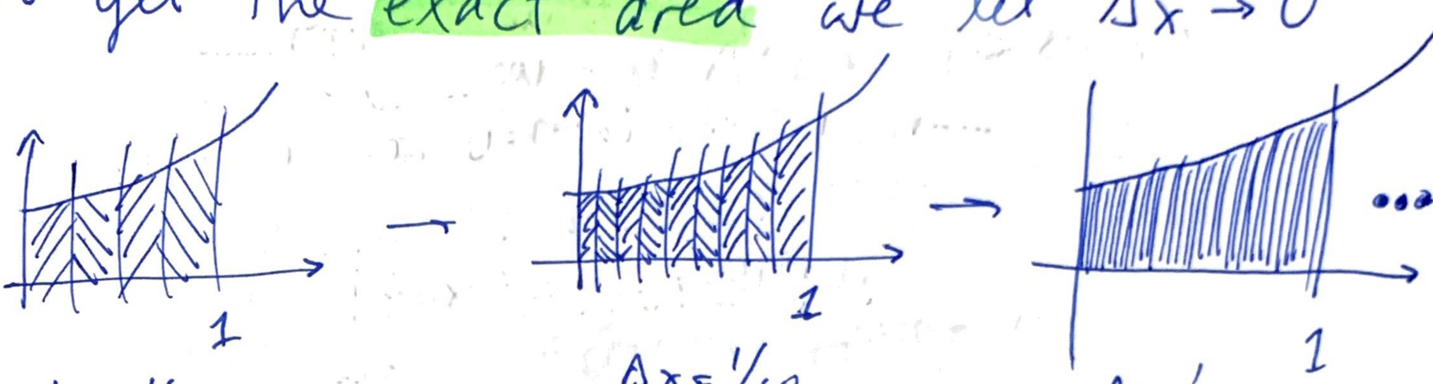
$$= \left(\frac{x^3}{3} + x\right)\Big|_0^1$$

$$= \left(\frac{1^3}{3}+1\right) - \left(\frac{0^3}{3}+0\right)$$

$$= \boxed{\frac{4}{3}} \approx 1.3333$$



To get the **exact area** we let $\Delta x \to 0$



$\Delta x = 1/4$ $\qquad$ $\Delta x = 1/10$ $\qquad$ $\Delta x = \frac{1}{100}$

• Integral Def: $\boxed{I = \lim_{\Delta x \to 0} \sum f(x) \cdot \Delta x}$ exact.

$$\boxed{\lim_{N \to \infty} \sum_{n=1}^{N} f(x_n) \cdot \underbrace{\frac{(b-a)}{N}}_{\Delta x}}$$

N = # of strips.

• Notation $$\boxed{\int_{x=a}^{x=b} f(x)\,dx \equiv \lim_{\Delta x \to 0} \sum \underbrace{f(\overset{(x_0 + n\Delta x)}{x}) \cdot \Delta x}} \quad \boxed{4.2}$$

"integral"

$\boxed{\infty \cdot 0}$ $\quad$ $\underline{\underline{\infty \text{ many}}}$ of $\underline{\underline{\infty\text{-ly thin}}}$ strips

$0 + 0 + 0 + \cdots$

• The mechanics, Details, are to be discussed soon.

$\boxed{0/0}$

Recall

Diff'n: $$f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right) \to \text{"}\frac{0}{0}\text{"}$$

= finite ratio

Review : Sum formulas

- From Pre-calc: We showed that by using proof-by-induction and other direct methods, that:
  - $\boxed{1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}}$

Ex Add 1 to 30 up:

$$1+2+3+\cdots+30 = \frac{30(3+1)}{2} = 15\cdot 4 = \boxed{60}$$

Euler as a kindergartener

$$1+2+3+\cdots+49+\underline{50}+51+\cdots+98+99+\underline{\underline{100}}$$

100

$$= 49\cdot 100 + 50 + 100$$

$$= 4900 + 150 = \boxed{5050}$$

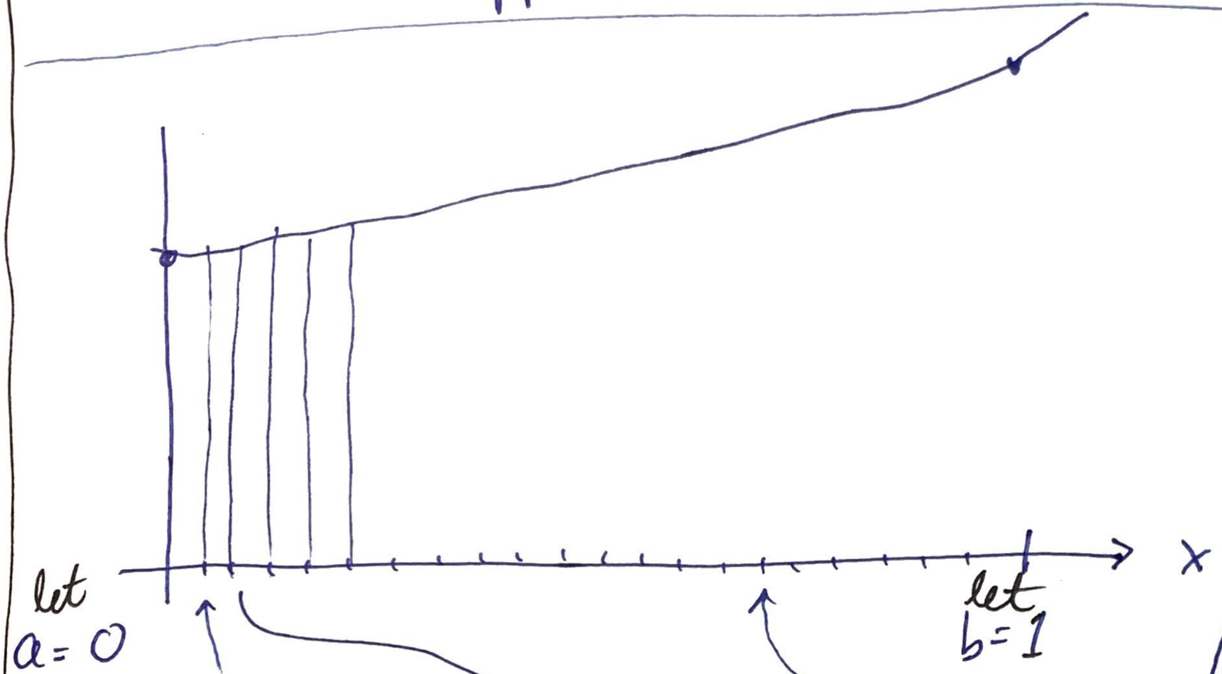
formula:

$$\frac{100(100+1)}{2} = 50\cdot 101 = \boxed{5050}$$

- $\boxed{1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$
- $1^3+2^3+3^3+\cdots+n^3 = \boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$

- Properties:
  - $\sum_{i=1}^{n} c = nc$
  - $\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right)$
  - $\sum_{i}^{n}(f_i \pm g_i) = \sum_{i}^{n} f_i \pm \sum_{i} g_i$

Ex Use $\sum_{1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to find an analytical formula for the sum of rectangles used to approx the area under $f(x) = x^2+1$ from 0 to 1



$$\boxed{\Delta x = \frac{b-a}{n}}$$

$x_1 = a + \Delta x$

$\quad = 0 + \left(\frac{b-a}{n}\right)$

$x_2 = a + 2\Delta x$

$x_3 = a + 3\Delta x$

$x_i = a + i\Delta x$

- since $a = 0$ and $b = 1$

$$x_i = 0 + i\left(\frac{1-0}{n}\right)$$

$$\boxed{x_i = i/n}$$ only for 0 to 1

$$\boxed{\Delta x = \frac{1-0}{n} = \frac{1}{n}}$$

Ex cont. 8

- Now $f(x_i)$ becomes $\left(\frac{i}{n}\right)^2+1$ since $f=x^2+1$

let R_n be defined as $\sum_{i=1}^{n} f(x_i)\Delta x$ { Riemann Sum

then

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x$$

$$R_n = \left[\left(\frac{1}{n}\right)^2+1\right]\left(\frac{1}{n}\right) + \left[\left(\frac{2}{n}\right)^2+1\right]\left(\frac{1}{n}\right) + \cdots + \left[\left(\frac{n-1}{n}\right)^2+1\right]\cdot\frac{1}{n} + \left[\left(\frac{n}{n}\right)^2+1\right]\frac{1}{n}$$

$$= \frac{1}{n}\left\{\frac{1^2}{n^2}+1 + \frac{2^2}{n^2}+1 + \frac{3^2}{n^2}+1 + \cdots + \frac{(n-1)^2}{n^2}+1 + \frac{n^2}{n^2}+1\right\}$$

$$= \frac{1}{n}\left\{\frac{1^2+2^2+3^2+\cdots+(n-1)^2+n^2}{n^2} + n\cdot 1\right\}$$

$$= \frac{1}{n}\left\{\frac{n(n+1)(2n+1)/6}{n^2} + n\right\}$$

$$\boxed{R_n = \frac{n(n+1)(2n+1)}{6n^3} + 1}$$

{ the approximation of the area (RHS) under $f(x)=x^2+1$ in $[0,1]$

- $R_4 = \frac{4(4+1)(2\cdot 4+1)}{6\cdot 4^3}+1 = \frac{45}{96}+1 = \frac{141}{96} = \boxed{\frac{47}{32}}$ ✓

- $R_{1000} = \frac{\cancel{1000}(1001)(2001)}{6\cdot\cancel{1000}\cdot 1000\cdot 1000} + 1 = 1.3338335$

Ex What if $n \to \infty$

exact area is $\lim_{n\to\infty} R_n$

$$A = \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3} + 1\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n}\right)\cdot\left(\frac{n+1}{n}\right)\cdot\left(\frac{2n+1}{n}\right)\cdot\frac{1}{6} + 1\right]$$

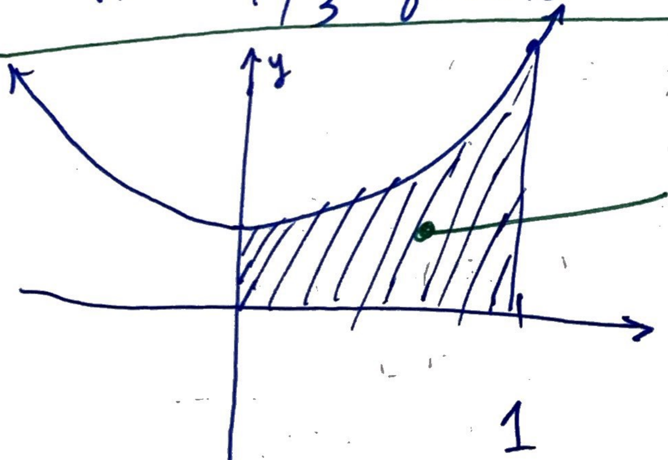
$$= \lim_{n\to\infty}\left[(1)\cdot\left(1+\cancelto{0}{\tfrac{1}{n}}\right)\cdot\left(2+\cancelto{0}{\tfrac{1}{n}}\right)\cdot\frac{1}{6} + 1\right]$$

$$= \frac{1\cdot 1\cdot 2}{6} + 1$$

$$= \frac{2}{6} + 1 = \frac{1}{3} + 1 = \boxed{\frac{4}{3}}$$

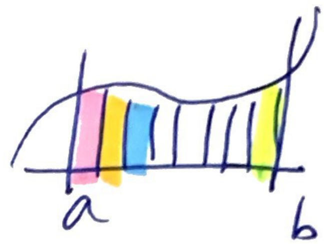
exact answer for ∞ many strips

The exact area under $f(x) = x^2 + 1$ from $x = 0$ to 1 is $4/3$ sq. units



area $= \frac{4}{3}$ sq. units.

⊗ General Definition



- width of $\boxed{\Delta x = \frac{b-a}{n}}$

- Subintervals $[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3] \cdots [x_{n-1}, x_n]$

- locations:

$$x_0 = a$$
$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$\vdots$$
$$x_n = a + n\Delta x$$

- heights

$$f(x_0) =$$
$$f(x_1) =$$
$$f(x_2) =$$
$$\vdots$$
$$f(x_n) =$$

- areas $f(x_i) \cdot \Delta x$

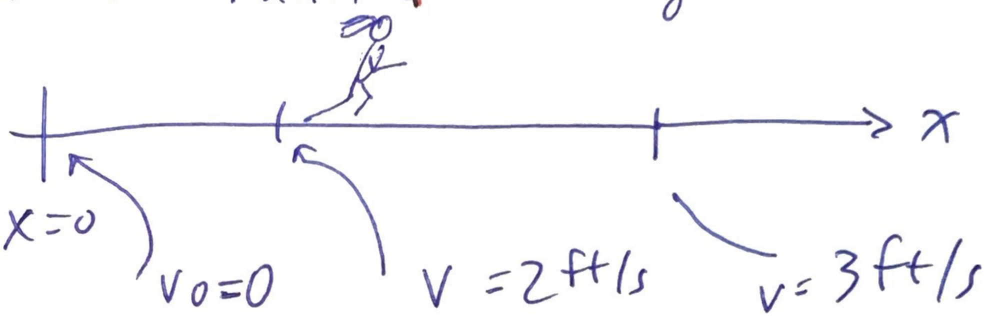
→ RHS $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

→ LHS $L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$

as $n \to \infty$ $\underbrace{R_n \to L_n}_{\text{Exact Area}}$

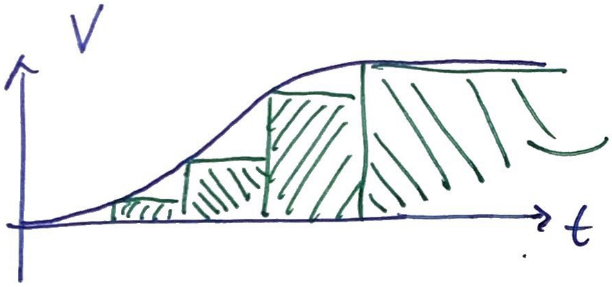
Application : Distance Covered by Runner

- Consider a runner running in the (+) x-direction



- Distance : $\boxed{D = V \cdot t}$

if v changes with time we need to use approximations, consider the V vs t plot:

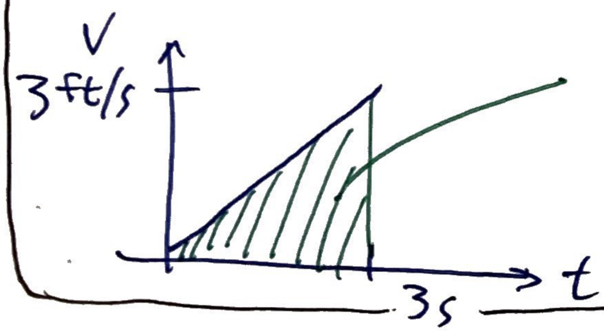


Area = height · Δx

$D = V \cdot \Delta t$

⇒ So area = Distance covered

Ex A runner acc'lt uniformly : What is the distance covered during "ramp-up"



Dist = area of a "v vs. t" plot:

$= \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}(3s)(3 ft/s) = 4.5 \text{ feet}$

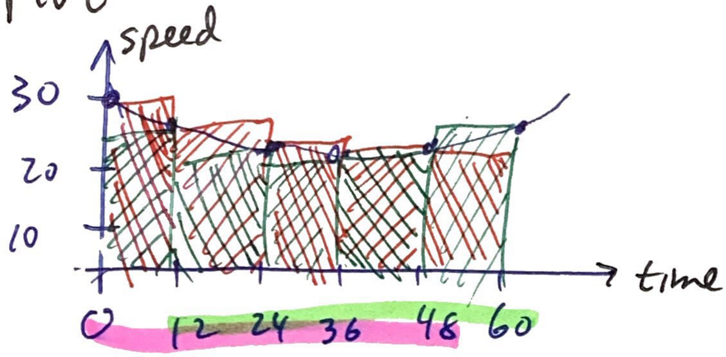
Ex The following table records a car's speed at uniform times. How far has the car moved?

- Table

$\Delta t = 12$, 12, 12, 12, 12

t	0	12	24	36	48	60	sec
V	30	28	25	22	24	27	m/s

- Plot



- LHS approx $= 30(12) + 28(12) + 25(12) + 22(12) + 24(12)$

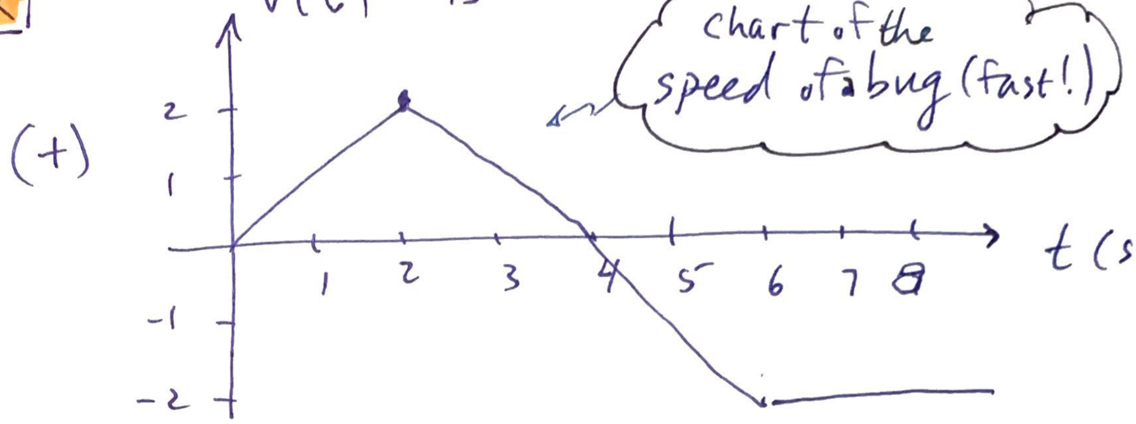
$= \boxed{1548\,m}$

- RHS approx $= 28(12) + 25(12) + 22(12) + 24(12) + 27(12)$

$= \boxed{1512\,m}$

Using a graph of $v(t)$ to compute distance.

EX

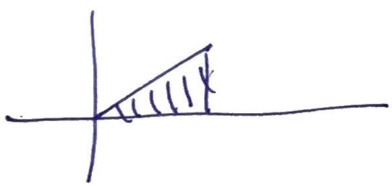


1-Dim motion

(a) given the above chart, at what time does the bug reverse direction? $\boxed{v(t) @ t = 4s}$

(b) after 2 sec how far has the bug travelled

$d = v \cdot t$ = area under curve between 0 & 2s



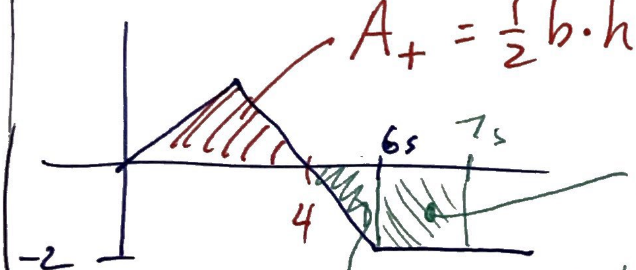
$d = \frac{1}{2}$ base · height

$= \frac{1}{2}(2 - 0\,s) \cdot (2\,ft/s) = \boxed{2\,ft}$

(c) When (time) has the bug reached the starting point?

Here we need to know when (+) area is equal to (−) area.

$A_+ = \frac{1}{2} b \cdot h = \frac{1}{2}(4 - 0_s) \cdot 2\frac{ft}{s} = \underline{\underline{4\,ft}}.$



$A = -2\,ft$

$A_- = \frac{1}{2} b \cdot h = \frac{1}{2}(6 - 4_s)(-2\frac{ft}{s}) = -2$ ft back @ $t = 6s$

we need another −2 ft so at $t = 7s$