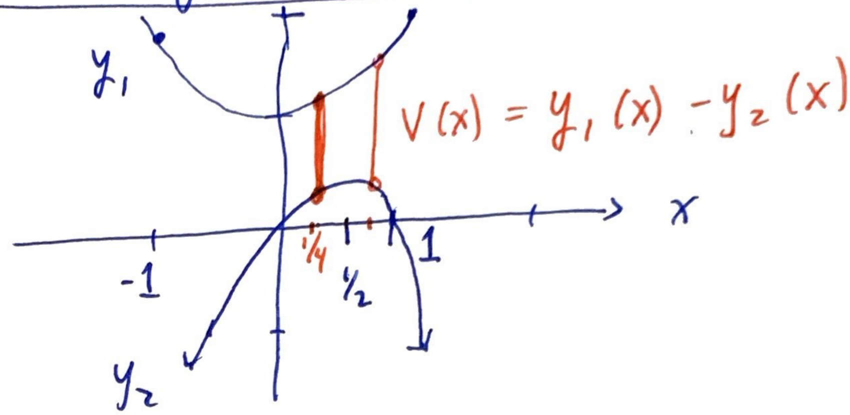
$\boxed{3.7}$ Optimization Problems (Word Problems)

Strategy

1. Read & Understand the problem
2. Draw diagrams, graph if possible
3. List variables, constants, etc. Create variables if needed to solve
4. Decide which governing equations to use
5. Substitute variables to simplify the problem to a single eqn and one dependent variable
6. Use extrema tools to optimize the eqn.
7. Answer the question

Ex Find the minimum vertical distance between two parabolas: $y_1 = x^2+1$, $y_2 = x - x^2$

- Diagram



- x = independ. var , V = dependent

- $V(x) = y_1(x) - y_2(x)$

$$= (x^2+1) - (x - x^2)$$

$$\boxed{V(x) = 2x^2 - x + 1}$$ minimize this eqn

- $V'(x) = 4x - 1$

$0 = 4x - 1 \rightarrow$ $\boxed{x = 1/4 \text{ critical point}}$ absolute min.

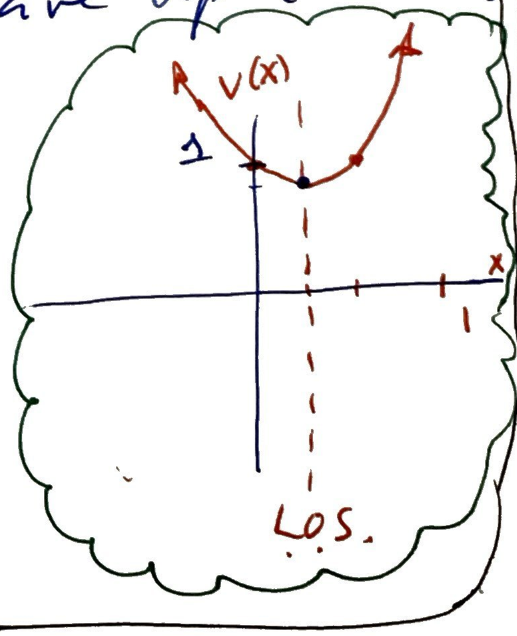
$V''(x) = 4 > 0$ so concave up (minimum)

- $V\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} + 1$

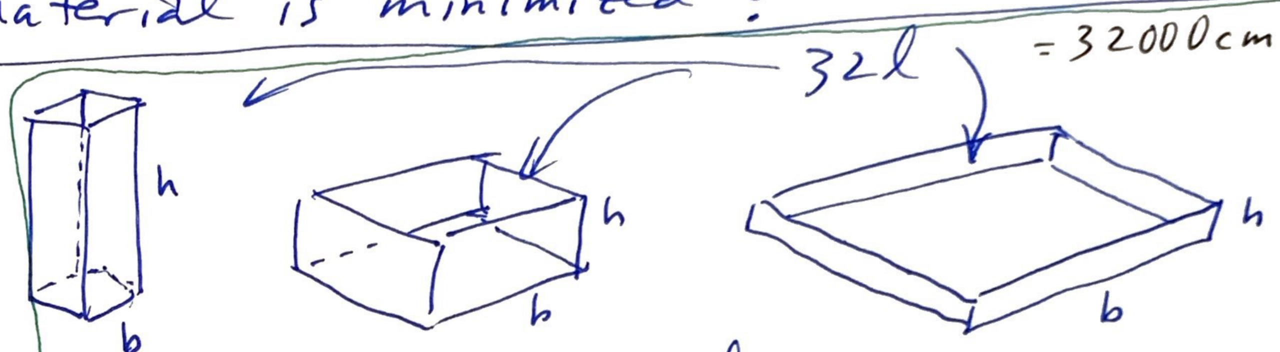
$$= \frac{1}{8} - \frac{2}{8} + \frac{8}{8}$$

$$= \boxed{7/8}$$ most min. vertical distance between curves.

ans.



Ex An open top box must contain 32 l of liquid. The base must be square. What should the dimensions be of the box so that the construction material is minimized?



Q: which uses less material to construct?

h = height, b = base, S = surface area

$\underline{S}$ = 4 sides + 1 base = $\underline{4hb + b^2}$

* We desire to minimize S * $\left\{ \begin{array}{l} \text{we need a relationship} \\ \text{between } b \text{ \& } h \end{array} \right.$

Use volume to connect b & h:

$$V = \text{Base} \cdot \text{height}$$

fixed amount $\longrightarrow V_0 = b^2 h \;\rightarrow\;$ Solve for $b = \sqrt{V_0/h}$

- Substitution:

$$S(b,h) = 4hb + b^2$$

$$S(h) = 4h\left(\frac{V_0}{h}\right)^{1/2} + \frac{V_0}{h}$$

$$\boxed{S(h) = 4\sqrt{V_0}\, h^{1/2} + V_0 h^{-1}}$$

minimize this

• Diff't $S'(h) = 4\sqrt{V_0}\,\frac{1}{2}h^{-1/2} + (V_0)(-1h^{-2}$

• Set to 0: $0 = 2\sqrt{V_0}/\sqrt{h} - \frac{V_0}{h^2}$

• Solve: $\left(\frac{2\sqrt{V_0}}{\sqrt{h}} = \frac{V_0}{h^2}\right)^2 \to \frac{4V_0}{h} = \frac{V_0^2}{h^4}$

$$\Rightarrow 4\cancel{V_0}h^{\cancel{4}\,3} = V_0^{\cancel{2}}\cancel{h}$$

$$4h^3 = V_0 \to \boxed{h = \sqrt[3]{V_0/4}}$$

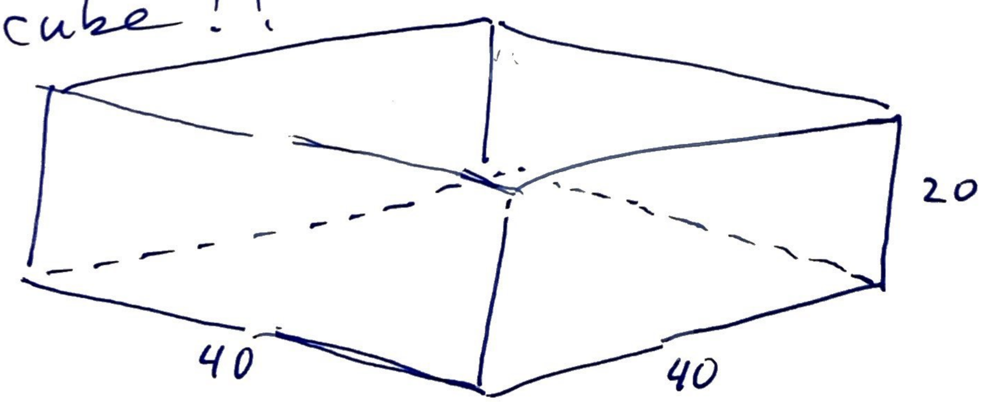
we also need b: $V_0 = b^2h \to \boxed{b = \sqrt{\frac{V_0}{h}}}$

• answer the questn

$$\to h = \sqrt[3]{\frac{32000}{4}} = \sqrt[3]{8000} = \sqrt[3]{2^3\cdot 10^3} = \boxed{20}$$

$$\to b = \sqrt{\frac{3200\cancel{0}}{2\cancel{0}}} = \sqrt{1600} = \sqrt{4^2\cdot 10^2} = 4\cdot 10 - 40$$

A $\frac{1}{2}$ cube!!



* Business Application

- $C(x)$ = cost function, it is the function that yeilds the cost of having produced x widgets {cars, phones, pencils, drugs}

- $C'(x)$ = marginal cost function, it is the rate of change of the cost function but it is also an approximation of the unit cost $\dfrac{dC(x)}{dx} \approx \dfrac{\Delta C}{\Delta x}$.

- $p(x)$ = sell price per unit that the company charges for the next unit AFTER the x^{th} unit is produced

- $R(x)$ = total revenue {cash intake} after selling x units

sell price per unit

$$\boxed{R(x) = x \cdot p(x)}$$

number of units sold

- $R'(x)$ = marinal rev. funct, the revenue/unit f selling the x^{th} unit. $\approx \Delta R / \Delta x$

- $P(x)$ = total profit received after selling x units.

$\Longrightarrow$ $P(x) = R(x) - C(x)$

$$\boxed{P(x) = x\,p(x) - C(x)}$$

- $P'(x)$ = marginal total profit, an approximation of the unit profit after we have sold x units.

- Average Profit/unit $= \dfrac{P(x)}{x}$

- Average Cost/unit $= \boxed{\bar{C}(x) = \dfrac{C(x)}{x}}$

Ex Show that if the ave. cost is at a minimum then the marginal cost is equal to the ave. cost.

- ave cost $= c(x) = \dfrac{C(x)}{x}$
- marginal cost $= C'(x)$

If ave cost = minimum

then $\boxed{C'(x) \overset{?}{=} \dfrac{C(x)}{x}}$ show this is the case

- ave cost = minimum

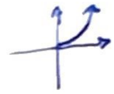
we need $\dfrac{d\left(C(x)/x\right)}{dx} = 0$

Apply the qoutient rule

$$\frac{d\left(\frac{C(x)}{x}\right)}{dx} = \frac{C' \cdot x - (x)' \cdot C(x)}{x^2}$$

$$0 = \frac{C' \cdot x - 1 \cdot C}{x^2}$$

$$\Rightarrow \boxed{C' = \frac{C}{x}} \quad \text{Q.E.D.}$$

EX let $C(x) = 16{,}000 + 200x + 4x^{3/2}$ 

startup costs

unit cost

maintanence costs.

(a) Find the cost, ave cost & marginal cost after producing the 1000th unit.

• $C(x)\Big|_{1000} = 16000 + 200(1000) + 4\sqrt{(1000)^3}$

$10^{3\cdot 3} = 10^9$

$= 16000 + 200000 + 40000\sqrt{10}$

$= \boxed{\$342{,}491}$

• ave cost $\dfrac{C(x)}{x} = \dfrac{C(1000)}{1000} = \boxed{\$342.49/\text{unit}}$

• marg. cost $= C'\Big|_{1000} = 0 + 200 + 4\cdot\frac{3}{2}x^{1/2}$

$= 200 + 6\sqrt{1000} \approx \boxed{\$389.74/\text{unit}}$

(b) What production level minimizes the average cost?

• ave cost $\dfrac{C(x)}{x}$

• min is when $\dfrac{d\left[C(x)/x\right]}{dx} = 0$ $\Rightarrow$

EX Cont. $C(x) = 16000 + 200x + 4x^{3/2}$

- $\frac{C(x)}{x} = 16000x^{-1} + 200 + 4x^{1/2}$

From the previous $\boxed{EX}$ we showed that the min. ave cost happens when it is equal to the marginal cost. @ the optimal production value, x.

So instead of minimizing $\frac{C(x)}{x}$ we set it equal to $C'(x)$:

- $C'(x) = 200 + 6\sqrt{x}$

But $\boxed{C'(x) = \frac{C(x)}{x}}$ at the minimum

$$\cancel{200} + 6x^{1/2} = 16000x^{-1} + \cancel{200} + 4x^{1/2}$$

$$\left[(6-4)x^{1/2} = 16000x^{-1}\right] * x$$

$$2x^{3/2} = 16000$$

$$x^{3/2} = 8000$$

$$x = \left(\sqrt[3]{(8000)}\right)^2$$

$$x = (20)^2$$

$\boxed{x = 400 \text{ units}}$ min. ave cost.

Ex (cont.) (9)

(c) What is the ave cost at the optimal prod. level?

At $x = 400$ units produced

$$C(400) = 16000 + 200 \cdot 400 + 4\left(\sqrt{400}\right)^3$$

$x^{3/2}$

$$= \$128{,}000$$

- ave cost $\frac{C(400)}{400} = \frac{128{,}000}{400} = \boxed{\$320/\text{unit}}$