3.5 Curve Sketching

Tools to date:

1. Domain / Range
2. Intercepts $\begin{cases} f(0) = y \text{ int} \\ f(x) = 0 \to x\text{-int} \end{cases}$
3. Symmetry: $f(-x) = f(x)$ even function
   $f(-x) = -f(x)$ odd funct.

   periodicity: $f(x+P) = f(x)$ period $= P$
4. Asymptotes:
   - $\lim_{x \to \pm\infty} f(x)$ HA
   - $\lim_{x \to a} f(x) = \pm\infty$ VA
   - long divide if necc'y to get an oblique asym.
5. Intervals of decreasing or increasing $f(x)$ by examing $f'(x) \lessgtr 0$

more...

6. Locate Extrema when $f' = 0$ @ $x = c$
and $f''(c) > 0$ or $f''(c) < 0$

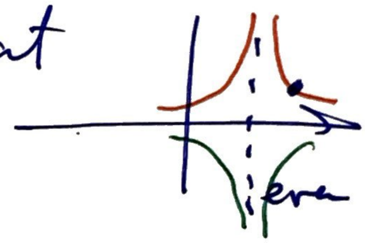
7. Study Concavity

$f''(x) > 0$ concave up

$f''(x) < 0$ concave down

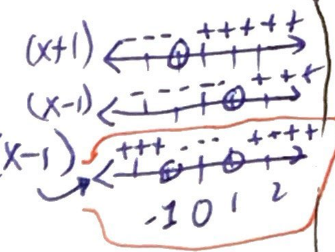
$f''(x) = 0$ at Inflection Points

8. Helper Point: to see where we start the graphing at



9. Start to sketch

Ex Sketch $y = \dfrac{x}{\sqrt{x^2-1}} = x(x^2-1)^{-1/2}$



- **Domain**: x^2-1 can't be $(-)$: $x^2-1 = (x+1)(x-1)$

  $D: \underline{\underline{(-\infty,-1)\cup(1,\infty)}}$

  avoid $\boxed{-1 \le x \le 1}$

  aka. $|x| \le 1$

- **y-int**: $f(0)$ but 0 is not in the domain

  x-int: $\dfrac{x}{\sqrt{x^2-1}} = 0 \Rightarrow \underline{\underline{x=0}}$ but that is not in the domain either!

  $\sqrt{x^2-1}$ $\ne 0$ outside of $|x|\le 1$

- **f(-x)**: $\underbrace{\dfrac{-x}{\sqrt{(-x)^2-1}}}_{f(-x)} = -\underbrace{\left(\dfrac{x}{\sqrt{x^2-1}}\right)}_{f(x)}$ $\underline{\underline{\text{odd}}}$ function

- **HA**: $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2-1}} \cdot \dfrac{1/x}{1/x} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1-\frac{1}{x^2}}} = \boxed{1}$

  $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-1}} \underset{u=-x}{=} \lim\limits_{u\to\infty} \dfrac{-u}{\sqrt{u^2-1}} = -\lim\limits_{u\to\infty} \dfrac{u}{\sqrt{u^2-1}} = \boxed{-1}$

- **VA**: we $\div$ by 0 @ $\boxed{x = \pm 1}$

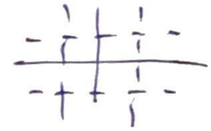
- **f'(x)** $= \left(x(x^2-1)^{-1/2}\right)' = x'(x^2-1)^{-1/2} + x\left[(x^2-1)^{-1/2}\right]'$

  $= \dfrac{1}{\sqrt{x^2-1}} + x\left[-\frac{1}{2}(x^2-1)^{-3/2}\cdot 2x\right]$

  $f' = \dfrac{-1}{(\sqrt{x^2-1})^3}$ **<0** in the dom

(cont)↓ $= \dfrac{1}{\sqrt{x^2-1}} - \dfrac{x^2}{(\sqrt{x^2-1})^3} = \dfrac{1}{\sqrt{x^2-1}}\dfrac{(\sqrt{x^2-1})^2}{(\sqrt{x^2-1})^2} - \dfrac{x^2}{(\sqrt{x^2-1})^3} = \dfrac{x^2-1-x^2}{(\sqrt{x^2-1})^3}$

Ex Cont. $= \frac{x}{\sqrt{x^2-1}}$, $f' = \frac{-1}{(\sqrt{x^2-1})^3}$

so $f'(x) < 0$ in $(-\infty,-1) \cup (1,\infty)$ 

- $f'(x)$ is not defined @ $x = \pm 1$ critical point. but $x=\pm1$ not in Domain

  $f' \neq 0$ anywhere in the domain

  So there are no critical points in the domain.

- $f'' = \left(-(x^2-1)^{-3/2}\right)' = -\left(-\frac{3}{2}\right)(x^2-1)^{-3/2-1}\cdot(2x)$

  $= \frac{3}{\cancel{2}}\cdot\cancel{2}x \, \frac{1}{(x^2-1)^{5/2}} = \frac{3x}{(\sqrt{x^2-1})^5}$

  → $f''<0$ for $x<0$, not in domain $(-\infty,-1)$ concave down

  → $f''>0$ for $x\in(1,\infty)$ concave up.

  → $f''\neq 0$ No I.P. since $x=0$ is not in the domain

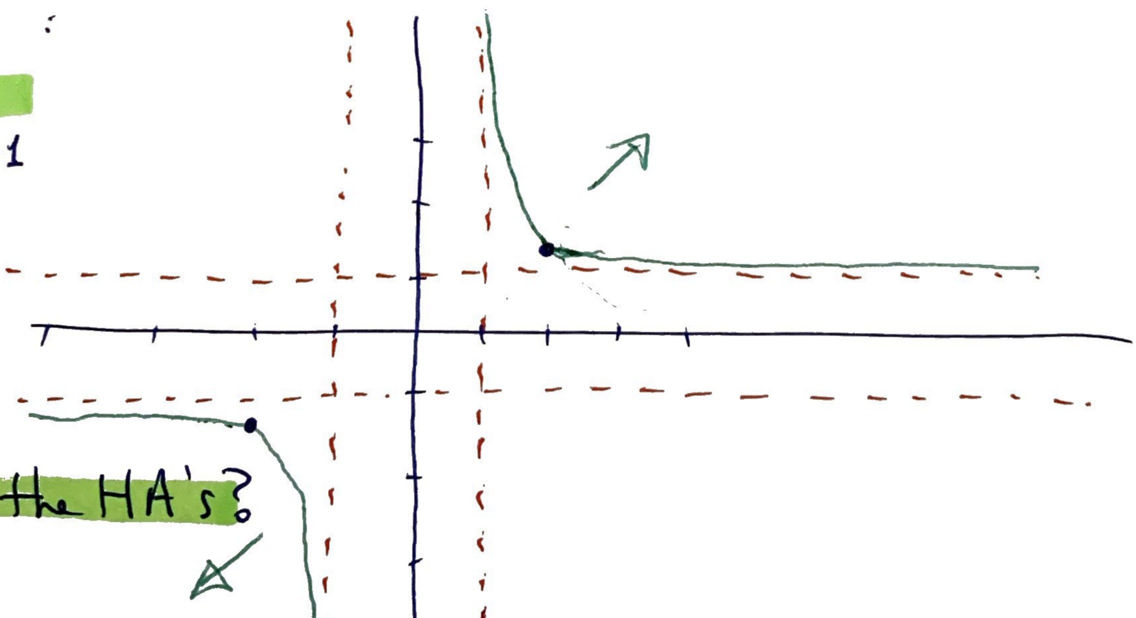
(*) Sketch it:

- Helper Points:

$f(2) = \frac{2}{\sqrt{2^2-1}} = \frac{2}{\sqrt{3}} > 1$

odd

$-2) = -\frac{2}{\sqrt{3}} < -1$



* Does f cross the HA's?

$f = \pm 1$

$\frac{x}{\sqrt{x^2-1}} = \pm 1 \Rightarrow x = \pm\sqrt{x^2-1} \Rightarrow x^2 = x^2-1 \Rightarrow 0 = -1$ * No, Does not cross HA

Ex Sketch $y = \dfrac{\ln x}{x^2}$

Reference: $\dfrac{d\ln(x)}{dx} = \dfrac{1}{x}$

- **Domain:** $(0, \infty)$
- **Int's:** $f(0)$ DNE $\boxed{\text{no y-int.}}$

$$\frac{\ln(x)}{x^2} = 0 \Rightarrow \ln(x) = 0 \quad \boxed{x = 1}$$

- no symmetry since we do not have $(-\infty, 0)$ in the domain.
- $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2} = 0$ To show this build a table

x	$\frac{\ln x}{x^2}$
1	
10	
100	
1000	0.000007
$\downarrow$ ∞	$\downarrow$ 0

So $\boxed{\text{HA: } y = 0}$

earlier exercise in chpt 2 we did a table $\boxed{\to -\infty}$

$$\lim_{x \to 0^+} \frac{\ln(x)}{x^2} = \lim \left(\frac{\ln(x)}{x}\right) \cdot \frac{1}{x} = -\infty \quad \boxed{\text{VA @ } x = 0}$$

- $f' = \dfrac{(\ln x)' x^2 - (\ln x)(x^2)'}{(x^2)^2} = \dfrac{\frac{1}{x} \cdot x^2 - (\ln(x))(2x)}{x^4} = \dfrac{1 - 2\ln(x)}{x^3}$

$\to f' > 0$ when $1 - 2\ln(x) > 0 \Rightarrow 1 > 2\ln(x) \Rightarrow \frac{1}{2} > \ln(x)$

raise both sides as exponents to base "e" $\left.\right\}$ $e^{1/2} > e^{\ln(x)}$

$e^{1/2} > x$

$\left\{ \begin{array}{l} g(x) > f(x) \\ \text{if your base function is increasing} \\ \text{the } a^{g(x)} > a^{f(x)} \end{array} \right\}$

$0 < \boxed{x < e^{1/2}}$ increasing

$\to f' = 0$ @ $x = e^{1/2}$

$\to f' < 0$ for $x \in (e^{1/2}, \infty)$

(cont.)

- $f'' = \left(\frac{1-2\ln(x)}{x^3}\right)'$

$$= \frac{(1-2\ln x)' x^3 - (1-2\ln x)(x^3)'}{x^6}$$

$$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

$$= \frac{-\frac{2}{x}\cdot x^3 - (1-2\ln x)\,3\cdot x^2}{x^6}$$

$$= \frac{-2x^2 - 3x^2 + 3\cdot 2\cdot x^2 \ln(x)}{x^6} = \underline{\underline{\frac{-5+6\ln(x)}{x^4}}}$$

$\rightarrow f'' > 0$ when $-5+6\ln x > 0$

$6\ln x > 5$

$\ln x > 5/6$

$e^{\ln x} > e^{5/6}$

$\boxed{x > e^{5/6}}$ concave up.

$\rightarrow f'' = 0$ @ $x = e^{5/6}$ (Inflection Point)

$\rightarrow f'' < 0$ when $\boxed{0 < x < e^{5/6}}$ concave down.

$e^{1/2} \approx \underline{\underline{1.7}}$

$e^{5/6} \approx 2.3$

Sketch