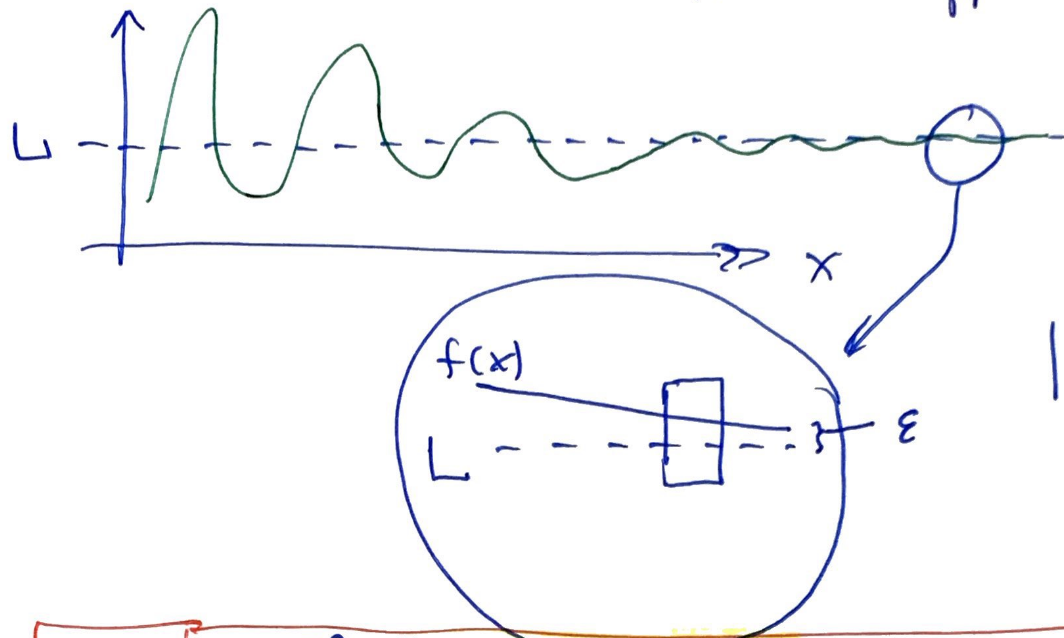
# 3.4 Limits at $\infty$



• consider a function that appears to approach the line $y=L$




$|f(x)-L| \leq \varepsilon$

**Def:** $\lim_{x\to\infty} f(x) = L$ if we are given an arbitrarily small $\varepsilon$ then we can find an increasing $x$ value where

$$|f(x)-L| \leq \varepsilon$$

**Ex:** let $\varepsilon = 0.1$ now if $f(x) \to L$ then you can return to me an $x$ value so that

$$|f-L| < 0.1$$

next let $\varepsilon = 0.01$, then you can find me an $x$-value such that

$$|f-L| < 0.01$$

$\vdots$

$< 0.00000000 \cdots 000000 1$

## Limit @ ∞ properties

Let $\lim_{x \to \infty} f(x) = F$ & $\lim_{x \to \infty} g(x) = G$

then

a) $\lim_{x \to \infty} [f \pm g] = F \pm G$

b) $\lim_{x \to \infty} [cf] = cF$

c) $\lim_{x \to \infty} [f(x) \cdot g(x)] = F \cdot G$

d) $\lim_{x \to \infty} \left[ f(x)/g(x) \right] = F/G$ if $G \neq 0$

e) $\lim_{x \to \infty} [f(x)]^n = F^n$

f) $\lim_{x \to \infty} [\sqrt[n]{f(x)}] = \sqrt[n]{F}$

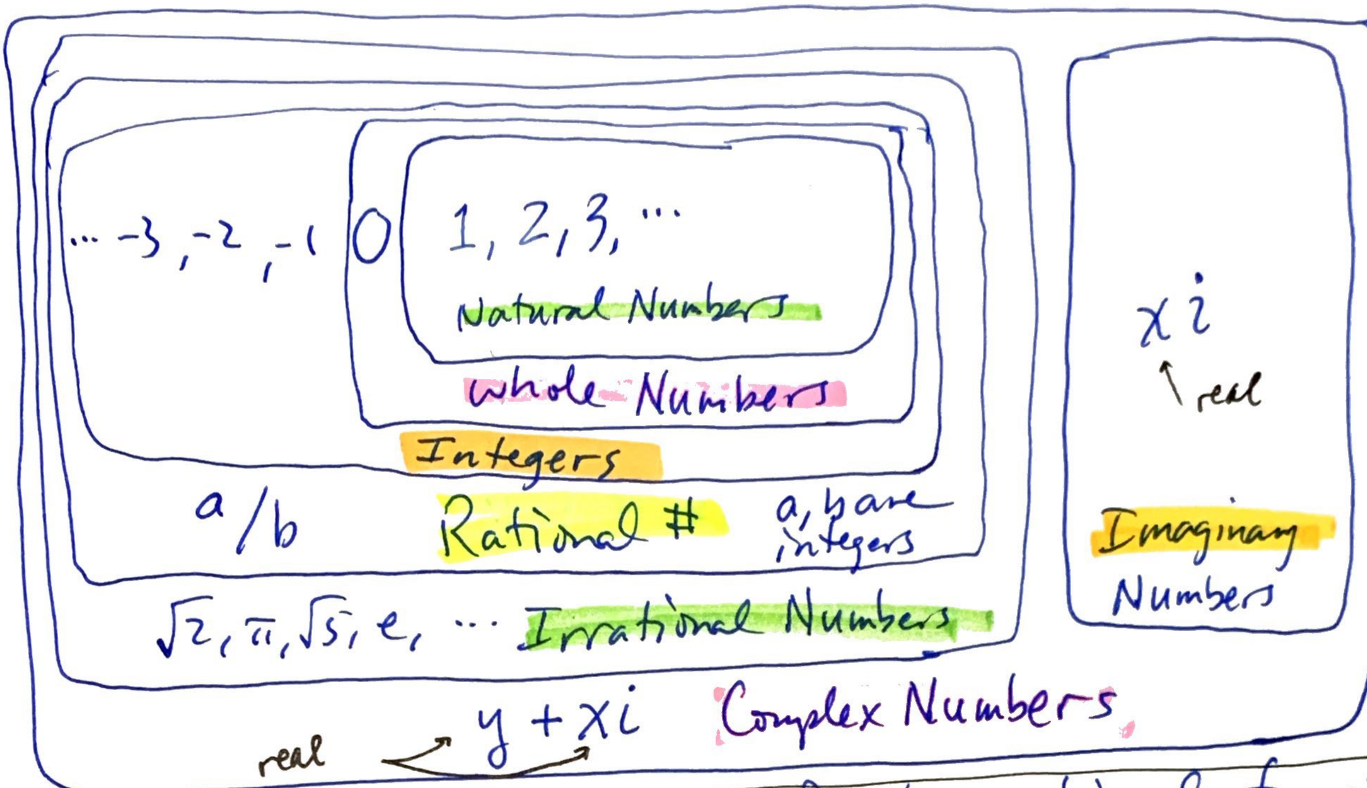
These repeat as $x \to -\infty$ also

## Comparison functions

(a) $\lim_{x \to \infty} \frac{1}{x^r} = 0$ if $r > 0$ & $r$ is rational #

(b) $\lim_{x \to -\infty} \frac{1}{x^r} = 0$ if $r > 0$ & $r$ is rational #

## Review of Number classes (types)




EX Evaluate the lim of the rational function

$$\lim_{x\to\infty}\left[\frac{4x^3+3x+1}{7x^3+2x^2+x}\right]\cdot\left(\frac{1/x^3}{1/x^3}\right)$$

- Form of $\frac{\infty}{\infty}$ so since there is no finite limit we can't use the Laws on page 2.
- multiplied top & bottom by $1/x^3$

$$\lim_{x\to\infty}\left[\frac{\frac{4x^3}{x^3}+\frac{3x}{x^3}+\frac{1}{x^3}}{\frac{7x^3}{x^3}+\frac{2x^2}{x^3}+\frac{x}{x^3}}\right]$$

$$=\lim_{x\to\infty}\left[\frac{4+3/x^2+1/x^3}{7+2/x+1/x^2}\right]$$

$$=\frac{\lim_{x\to\infty}(4+3/x^2+1/x^3)}{\lim(7+2/x+1/x^2)}=\boxed{\frac{4}{7}}$$

$\frac{\text{deg top}=3}{\text{deg bot}=3}$

**Ex** $\lim_{x\to\infty} \left[\frac{4x^3+3x+1}{7x^4+2x^2+x}\right] \cdot \left(\frac{1/x^4}{1/x^4}\right)$

$$= \lim_{x\to\infty} \left[\frac{4/x + 3/x^3 + 1/x^4}{7 + 2/x^2 + 1/x^3}\right]$$

$$= \frac{\lim_{x\to\infty} \left[4/x + 3/x^3 + 1/x^4\right]}{\lim_{x\to\infty} \left[7 + 2/x^2 + 1/x^3\right]} = \frac{0}{7} = \boxed{0}$$

This happens because deg top < deg bot

**Ex** $\lim_{x\to\infty} \left[\frac{4x^4+3x+1}{7x^3+2x^2+x}\right] \cdot \left(\frac{1/x^4}{1/x^4}\right)$

$$= \lim_{x\to\infty} \frac{\left[4+3/x^3+1/x^4\right] \to 4}{\left[7/x + 2/x^2 + 1/x^3\right] \to 0} = \frac{F}{G} = 0 !!$$

here $\frac{4}{0}$ so this limit is undefind "$\infty$"

**Summary:** • $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \frac{a_n}{b_n}$ if $\deg(p) = \deg(q)$

and $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$
$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x + b_0$

• $\lim_{x\to\infty} \left(\frac{p}{q}\right) = 0$ if $\deg(p) < \deg(q)$

• $\lim_{x\to\infty} \left(\frac{p}{q}\right) = \infty$ if $\deg(p) > \deg(q)$

- For the case of $\deg(p) > \deg(q)$ then the ratio approaches a <u>limit function</u>, which is the quotient of the long division of $p(x)/q(x)$

Nomenclature:

$$\frac{p(x)}{q(x)} = \boxed{l(x)} + r(x)$$

Dividend

Divisor

quotient

remainder

- $\boxed{\lim_{x\to\infty}\left(\frac{p(x)}{q(x)}\right) = l(x)}$ if $\deg(p) > \deg(q)$

{Note $\lim_{x\to\infty} l(x) = \infty$ }

**Recall from 5th grade:**

$$\frac{471}{9} \Rightarrow \begin{array}{r} 52 \text{ r } 3 \\ 9\,\overline{)\,471} \\ -45\phantom{1} \\ \hline 21 \\ -18 \\ \hline 3 \end{array}$$

$$\frac{471}{9} = 52 \text{ r } 3 = \boxed{52 + \frac{3}{9}} = 52\tfrac{1}{3}$$

Boss's answer

we stop here and use the quotient as an oblique HA

$\boxed{\text{EX}}$ Find the oblique asymptote of

$$y = \frac{2x^5 - 3x^4 + x^3 - x^2 + 11x - 2}{x^2 + 7}$$

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Since $\deg(\text{top}) > \deg(\text{bottom})$ we have an "improper rational funct."

- We long divide

$$
\begin{array}{r|l}
 & 2x^3 - 3x^2 - 13x + 20 \quad \text{r. } 102x - 142 \\
\hline
x^2 + 0x + 7 & 2x^5 - 3x^4 + x^3 - x^2 + 11x - 2 \\
 & -(2x^5 + 0x^4 + 14x^3) \\
\hline
 & -3x^4 - 13x^3 - x^2 \\
 & -(-3x^4 + 0x^3 - 21x^2) \\
\hline
 & -13x^3 + 20x^2 + 11x \\
 & -(-13x^3 + 0x^2 - 91x) \\
\hline
 & 20x^2 + 102x - 2 \\
 & -(20x^2 + 0x + 140) \\
\hline
 & 102x - 142
\end{array}
$$

- So "y" can be rewritten to be:

$$\boxed{y = 2x^3 - 3x^2 - 13x + 20 + \left(\frac{102x - 142}{x^2 + 7}\right)}$$

proper rational expression

- Desmos: $\begin{cases} \text{Original "y" plotted in one row} \\ 2x^3 - 3x^2 - 13x + 20 \text{ in the second row} \end{cases}$

EX Study $y = \dfrac{2x^2+1}{\sqrt{x^4-4}}$ at $\pm\infty$

Note that $x^4-4$ approaches $x^4$ as $x \to \pm\infty$

So $\sqrt{x^4-4} \to \sqrt{x^4}$ or $x^2$

then as $x \to \pm\infty$ $y \to \dfrac{2x^2}{x^2} = \boxed{2}$

- techically (justification)

$$y = \frac{2x^2+1}{\sqrt{x^4-4}} \cdot \left(\frac{1/x^2}{1/x^2}\right)$$

use in reverse

$$\frac{1}{\sqrt{\frac{1}{x^4}}} = \frac{1}{1/x^2}$$

$$y = \frac{2 + 1/x^2}{\sqrt{\frac{x^4-4}{x^4}}}$$

b/c

$$\sqrt{x^4} = (x^4)^{\frac{1}{2}} = x^{4/2}$$
$$= x^2$$

$$y = \frac{2 + 1/x^2}{\sqrt{1 - 4/x^4}}$$

$$\lim_{x \to \infty} \left( \frac{2 + 1/x^2}{\sqrt{1 - 4/x^4}} \right)$$

$$= \frac{\lim (2 + 1/x^2)}{\lim \left(\sqrt{1 - 4/x^4}\right)}$$

$$= \frac{2}{1} = \boxed{2}$$

HA

Q: what happens near x=0 ? 

• Factor $x^4-4 = (x^2-2)(x^2+2)$

$$y = \frac{2x^2+1}{\sqrt{x^2-2}\,\sqrt{x^2+2}}$$

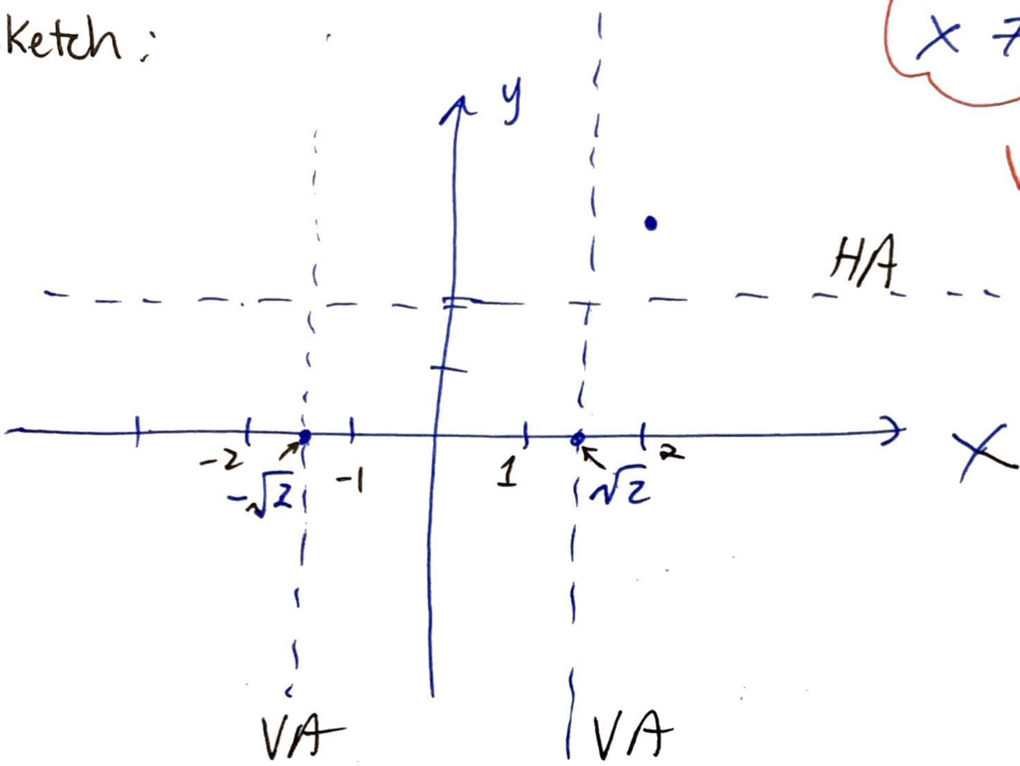
• VA

$x^2-2 \neq 0$

$x \neq \pm\sqrt{2}$ $\approx 1.4$

VA

• Pre-Sketch:



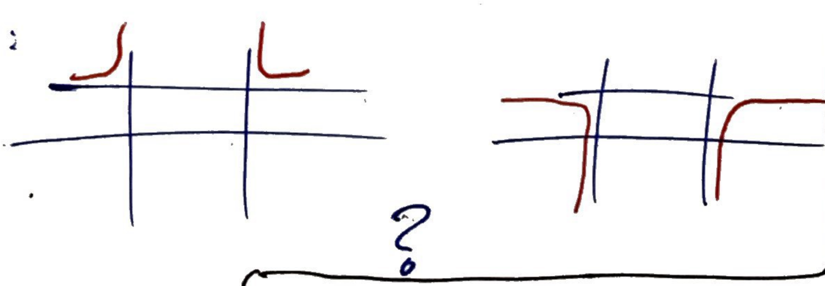

• Domain: $x^4-4 \not< 0$ b/c $\sqrt{-\#} \neq$ real

$$(x^2-2)\underbrace{(x^2+2)}_{(+)} \not< 0$$

$$\Rightarrow \quad x^2-2 \not< 0$$

$$x^2 \not< 2 \quad = \quad x \notin (-\sqrt{2}, \sqrt{2})$$

• Sketch choices:



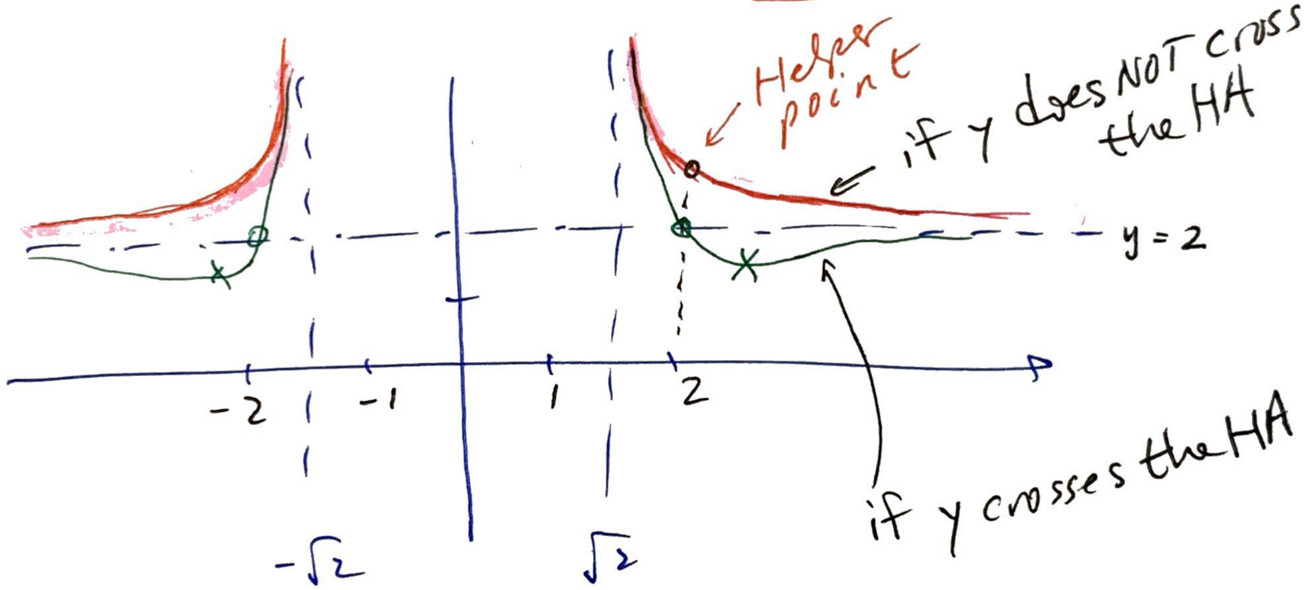
- Helper point @ $x=2$:

$$y = \frac{2x^2+1}{\sqrt{x^4-2}}\Bigg|_{x=2} = \frac{2^3+1}{\sqrt{16-2}} > 0 \ (+)$$

Wait, not usefull since HA is (+) also }

But, note that $y(2) = \frac{9}{\sqrt{14}} = \underline{\underline{3.7}}$ this is above the HA

This tells me that use " ⌟ ⌞ "




Q: Does the curve cross the HA?

A: Set $y=2$ and solve for $x$:

$$2 = \frac{2x^2+1}{\sqrt{x^4-4}} \rightarrow 4 = \frac{(2x^2+1)^2}{x^4-4}$$

$$\Rightarrow \cancel{4x^4} - 4 = \cancel{4x^4} + 4x^2 + 1$$

$$0 = 4x^2 + 5$$

$x \neq$ Real $\Longrightarrow$ No Crossing