3.3 B Rational Functions

- A **rational number** is a number that is created by forming a ratio of two whole numbers

$$a/b$$

Ex $13/17$, $13/99$, $1/9$

0.7647, $0.1313131\overline{3}$, $= 0.11111 = 0.\overline{1}$

- A **rational function** is the ratio of polynomials

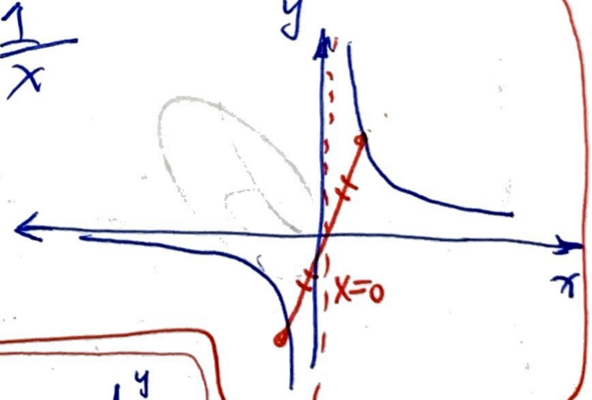
Ex $\dfrac{x+1}{x^2-3x+1}$ ← $p(x)$, ← $q(x)$

Let $\boxed{R(x) = \dfrac{f(x)}{g(x)}}$ be our rational function.

⊛ **Reciprocal function** $R(x) = \dfrac{1}{x}$

$f(-x) = -f(x)$

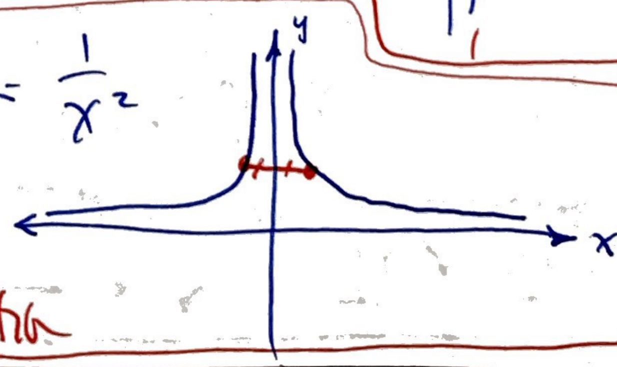
odd function



⊛ **Volcano function** $R(x) = \dfrac{1}{x^2}$

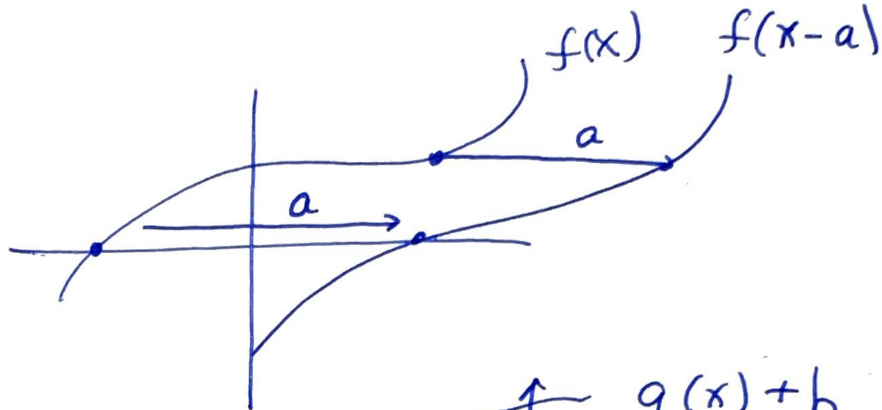
$f(-x) = f(x)$

even function

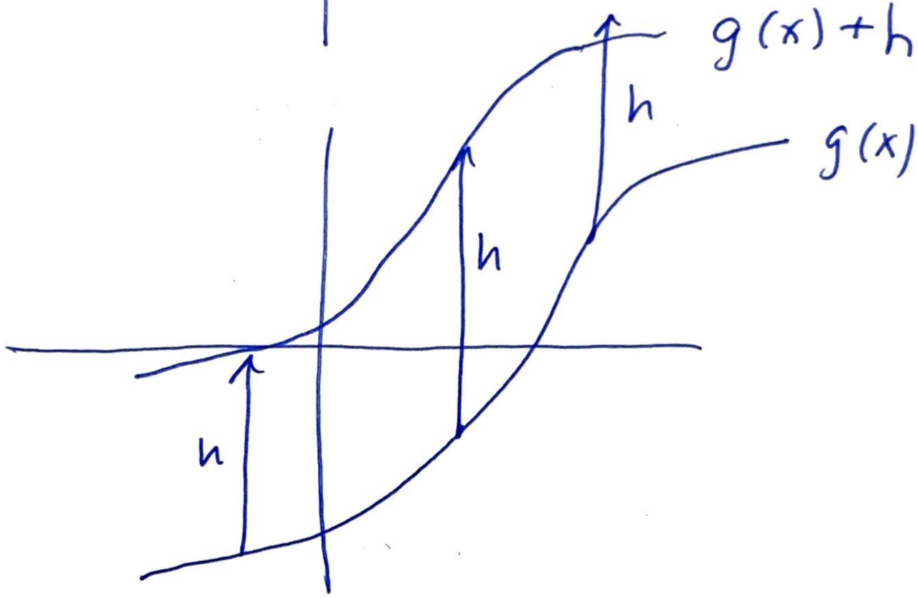


vertical asymptotes occure where the denominator vanishes (0)

⊛ Transformations

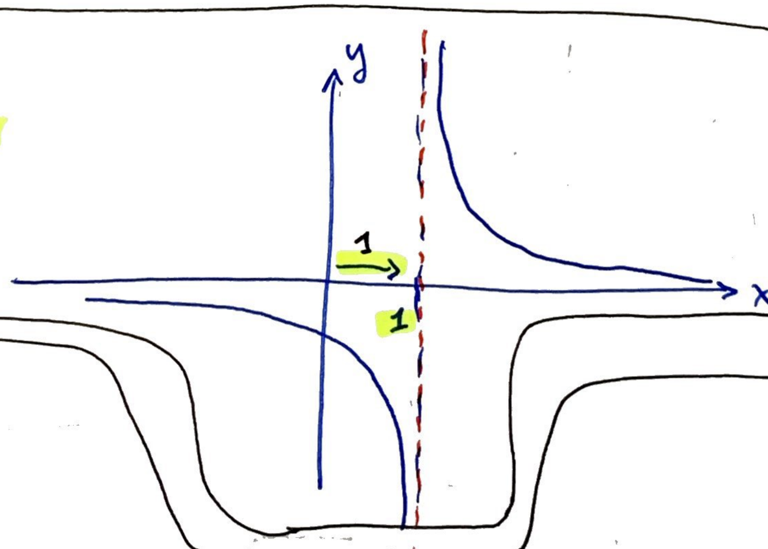


• Horizontal



• Vertical

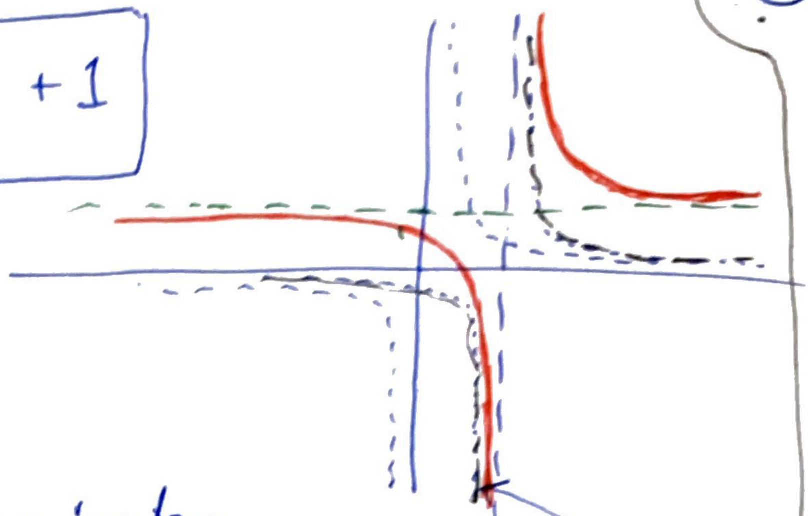
Ex $R(x) = \frac{1}{x-1}$



Ex $R(X) = \frac{1}{X^2} + 2$

Ex

• Together $\boxed{F(x) = \frac{1}{x-1} + 1}$



BTW: common denominator

$$F(x) = \frac{1}{x-1} + 1\left(\frac{x-1}{x-1}\right)$$

$$F(x) = \frac{1}{x-1} + \frac{x-1}{x-1} = \frac{1+x-1}{x-1} = \frac{x}{x-1}$$

So $\boxed{F(x) = \frac{x}{x-1}}$

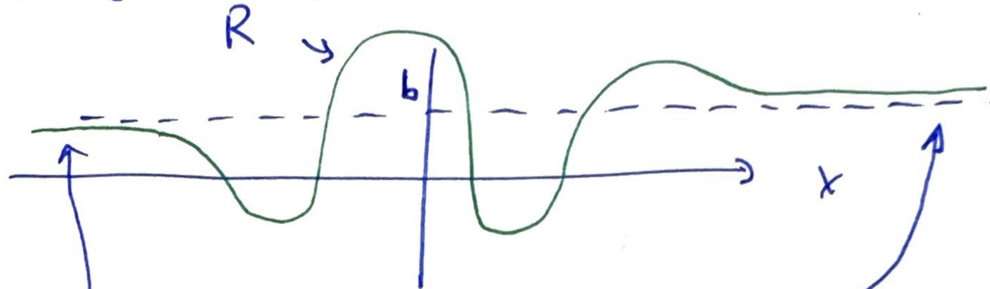
• consider $J(x) = \frac{x}{(x-1)^2}$. what does it's graph look like?

T.B.D.

(*) We now consider <u>graphing rational functions</u> while in their ratio form, not in any type of transformation form.

⊛ Horizontal Asymptotes

- H.A. occurs when, as $x \to \pm\infty$, the function approaches a fixed value.



$$\lim_{x\to\infty} R(x) = b^+$$

$$\lim_{x\to-\infty} R(x) = b^-$$

Q: How do we know what the HA is?
{ Note there is only one HA for rational expressions }

A: Depends on the powers of the numerator vs. the power of the denominator.

Consider $R(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x^1 + b_0}$

- if $n = m$: HA is $\boxed{y = a_n / b_n}$
- if $n < m$: HA is $\boxed{y = 0}$
- if $n > m$: HA is slant or oblique and requires long division to obtain.

Ex Consider $F(x) = \dfrac{3x^2+1}{2x^2-x+11}$, Find the H.A.

Deg Top = 2
Deg Bot = 2 $\Big\}$ $\boxed{y = \frac{3}{2}}$ is the H A

Ex Consider $G(x) = \dfrac{4x^2-x+12}{5x^3-2x^2+11x+1}$

Deg Top = 2
Deg Bot = 3 $\Big\}$ $\boxed{y=0}$ is the H.A.

- Before we study the oblique asymptotes lets further investigate these limits

EX For $F(x) = \dfrac{3x^2+1}{2x^2-x+11} \cdot \underbrace{\left(\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)}_{=1}$ ← we multiplied by a "magic one"

$$= \frac{\frac{3x^2+1}{x^2}}{\frac{2x^2-x+11}{x^2}}$$

$$= \frac{\frac{3x^2}{x^2}+\frac{1}{x^2}}{\frac{2x^2}{x^2}-\frac{x}{x^2}+\frac{11}{x^2}} = \frac{3+\cancel{1/x^2}^{\to 0}}{2-\cancel{1/x}^{\to 0}+\cancel{11/x^2}^{\to 0}}$$

- Consider the limit as $x \to \pm\infty$:

$$\lim_{x\to\infty} F(x) = \boxed{\frac{3}{2}}$$

EX

For $G(x) = \dfrac{4x^2 - x + 12}{5x^3 - 2x^2 + 11x + 1} \left(\dfrac{1/x^3}{1/x^3}\right)$ pick larger degree

$$= \dfrac{\dfrac{4x^2 - x + 12}{x^3}}{\dfrac{5x^3 - 2x^2 + 11x + 1}{x^3}}$$

$$= \dfrac{\dfrac{4x^2}{x^3} - \dfrac{x}{x^3} + \dfrac{12}{x^3}}{\dfrac{5x^3}{x^3} - \dfrac{2x^2}{x^3} + \dfrac{11x}{x^3} + \dfrac{1}{x^3}}$$

$$\frac{x^2}{x^3} = x^{2-3} = x^{-1} = 1/x$$

$$= \dfrac{\boxed{\cancel{4/x}} \to 0 - \cancel{1/x^2} \to 0 + \cancel{12/x^3} \to 0}{\boxed{5} - \cancel{2/x} \to 0 + \cancel{11/x^2} \to 0 + \cancel{1/x^3} \to 0}$$

Now take the limit as $x \to \pm\infty$

HA: $y = \dfrac{0}{5} = 0$

To address the case of $n \geq m$ { deg Top $\geq$ deg Bot }

we need to review long division.

- Classic Long Division, start by reviewing 5th grade.

EX $47/3$

improper fraction

$$\begin{array}{r} 15 \\ 3\overline{)47} \\ -3\downarrow \\ \hline 17 \\ -15 \\ \hline 2 \end{array}$$

$47/3 = 15 \text{ r } 2$

$$\boxed{47/3 = 15 + \frac{2}{3}} = 15\tfrac{2}{3}$$

proper fraction

EX Longdivide: $\dfrac{3x^3+2x^2-4x+1}{x-1}$ improper fraction

Form

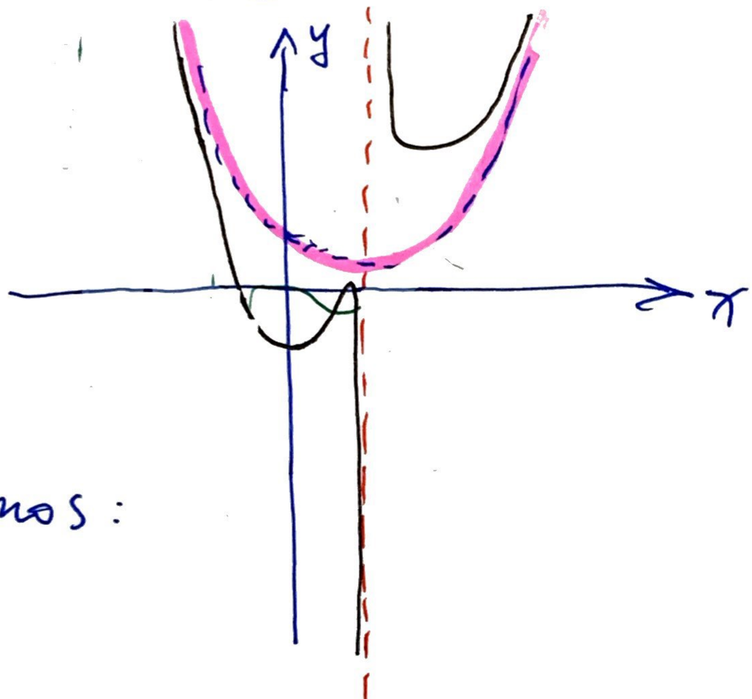
$$\begin{array}{r} 3x^2+5x+1 \\ x-1\overline{)3x^3+2x^2-4x+1} \\ -(3x^3-3x^2)\downarrow \\ \hline 5x^2-4x \\ -(5x^2-5x) \\ \hline x+1 \\ -(x-1) \\ \hline 2 \end{array}$$

So $$\boxed{\frac{3x^3+2x^2-4x+1}{x-1} = 3x^2+5x+1+\frac{2}{x-1}}$$

- Now lets find the HA of the previous example

$$\frac{3x^3+2x^2-4x+1}{x-1} = 3x^2+5x+1 + \cancel{\frac{2}{x-1}} \to 0$$

As $x \to \pm\infty$ this rational function approaches the parabola $3x^2+5x+1$ since the remainder term vanishes.



Desmos:

⊛ Procedures to Graph Rational functions

1. Find the y-intercept ((R(0)))
2. Factor Numerator and denominator
3. zeros of the R(x) occure when the numerator vanishes
4. Vertical asymptotes occur when the denominator vanishes.
5. Note the oddness or evenness of the VA.
6. Determine the horizontal / Slant / Oblique asymptote via long divisiL ((if needed))
7. Determine if the curve crosses the HA.
8. Start sketching
9. Use helper points
10. Use $f'(x)=0$ to get critical points and use $f''(x)$ to determin local max/min.
11. Refine sketching.

Sketch $f(x) = \dfrac{}{x^2-9}$

1. y-int: $f(0) = \dfrac{0}{0^2-9} = 0$ $\boxed{y=0}$

2. factor: $f(x) = \dfrac{x}{(x+3)(x-3)}$ $(x-3)^1$ $(x+3)^1$

3. zeros: $\boxed{x=0}$

4. VA: $x=-3$ odd $x=3$ odd

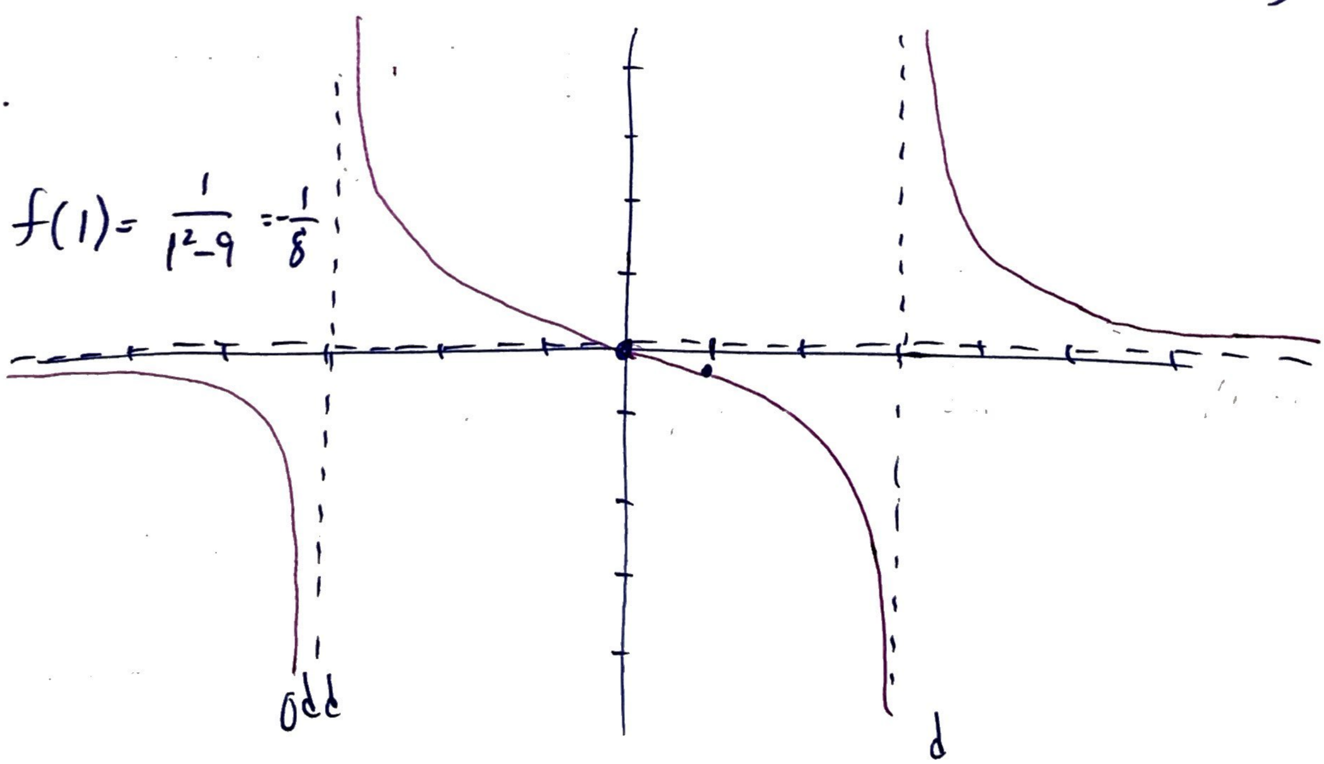
5. {odd  or  vs  or  even}

6. HA: deg. top < deg bot so $\boxed{y=0}$

7.  : $\boxed{f(x)=HA} \Rightarrow f(x)=0 \Rightarrow \boxed{x=0}$ same #3

8.

9. $f(1) = \dfrac{1}{1^2-9} = -\dfrac{1}{8}$



Sketch $f(x) = \dfrac{x^2 + x + 6}{x^2 - 10x + 24}$

1. $f(0) = \dfrac{0^2 + 0 + 6}{0^2 - 10\cdot 0 + 24} = \dfrac{6}{24} = \dfrac{1}{4}$ y-int: $\boxed{y = 1/4}$

2. $f(x) = \dfrac{(\ ?\ )(\ ?\ )}{(x-4)(x-6)}$ $x^2 + x + 6 = 0$

   $x = \dfrac{-(1) \pm \sqrt{1^2 - 4\cdot 1\cdot 6}}{2\cdot 1} = \dfrac{-1}{2} \pm \dfrac{\sqrt{-\#}}{2}$

   No Real Roots

3. zeros : None

4. VA: $x = 4$, $x = 6$

5. odd odd

6. HA: deg top $= 2$, deg bot $= 2$ so HA: $\underline{\underline{y = \frac{1}{1} = 1}}$

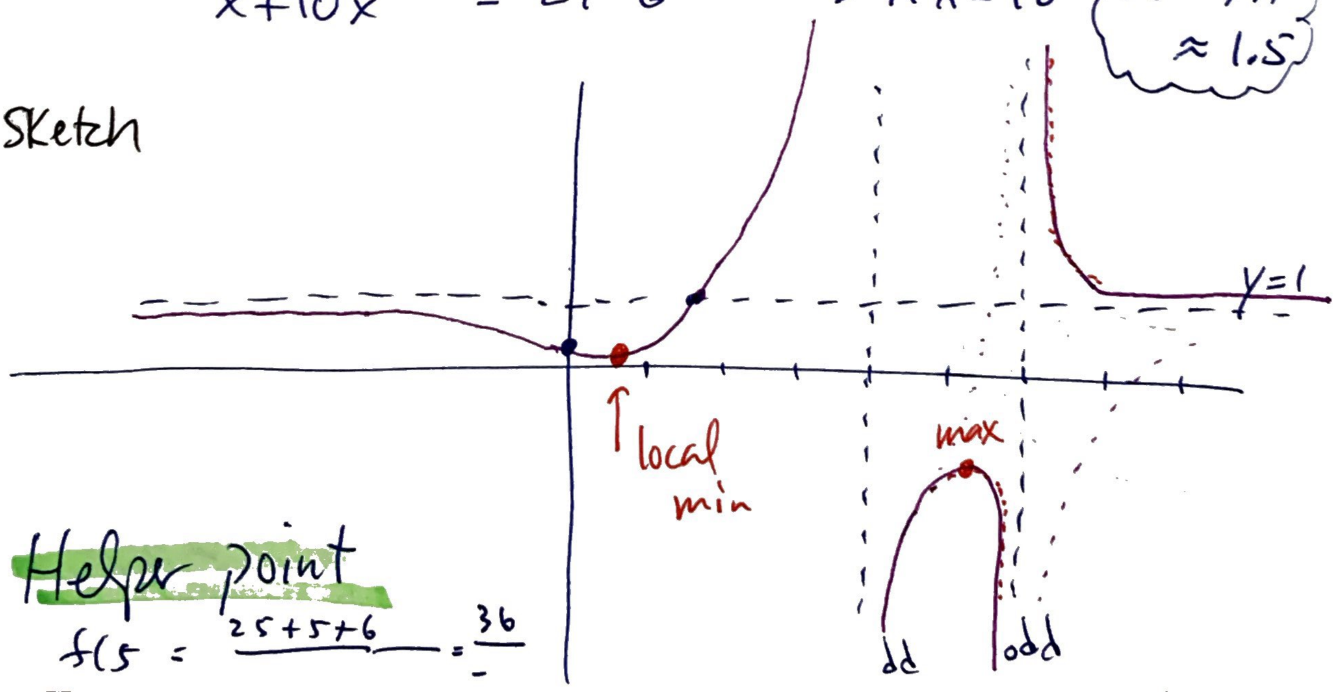
7. Crossing HA? I.E. where is $f(x) = 1$

   $\Rightarrow \dfrac{x^2 + x + 6}{x^2 - 10x + 24} = 1$ cross-multiply

   $\cancel{x^2} + x + 6 = \cancel{x^2} - 10x + 24$

   $x + 10x = 24 - 6 \rightarrow 11x = 18$ $x = 18/11 \approx 1.5$

8. Sketch



9. Helper point

   $f(5 = \dfrac{25+5+6}{} = \dfrac{36}{-}$

Next where does this curve max out, min out?
(local)

10. $$f'(x) = \frac{(x^2+x+6)'(x^2-10x+24) - (x^2+x+6)(x^2-10x+24)'}{(x^2-10x+24)^2}$$

$$= \frac{(2x+1)(x^2-10x+24) - (x^2+x+6)(2x-10)}{(\quad)^2}$$

foil

$$= \frac{2x^3-20x+48+x^2-10x+24-\{2x^3+2x^2+12x-10x^2-10x-60\}}{(\quad)^2}$$

$$\text{TOP} = 2x^3-20x+48+x^2-10x+24-2x^3-2x^2-12x+10x^2+10x+60$$

$$0 = 0x^3+9x^2-32x+132$$

$$0 = 9x^2-32x+132 \Rightarrow x =$$