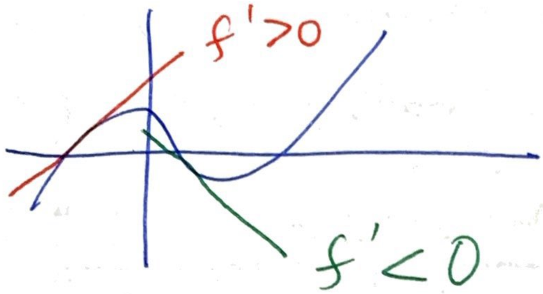
3.3 Using Derivatives to Determine the Shapes of curves

I What does f' say about f ?

- If $f'(x) > 0$ then f is increasing
  If $f'(x) < 0$ then f is decreasing



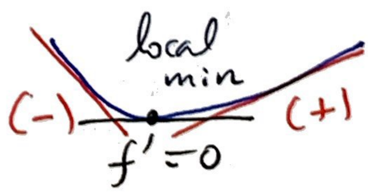
The derivative Test:

Suppose "c" is a critical value.

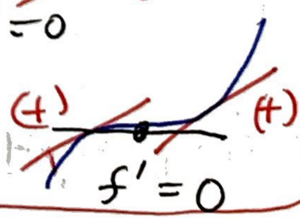
i.e. $f'(c) = 0$ or $f'(c)$ is undefined or $f'(c)$ D.N.E.

(a) If f' changes sign from (+) to (−) then $f(c)$ is a local max.

(b) If f' changes from (−) to (+) then $f(c)$ is a local min



(c) If f' around "c" does not change sign f has no local max or min

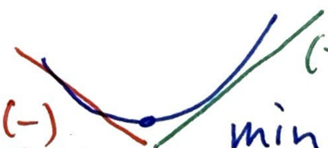


II Sketching steps (part 1)

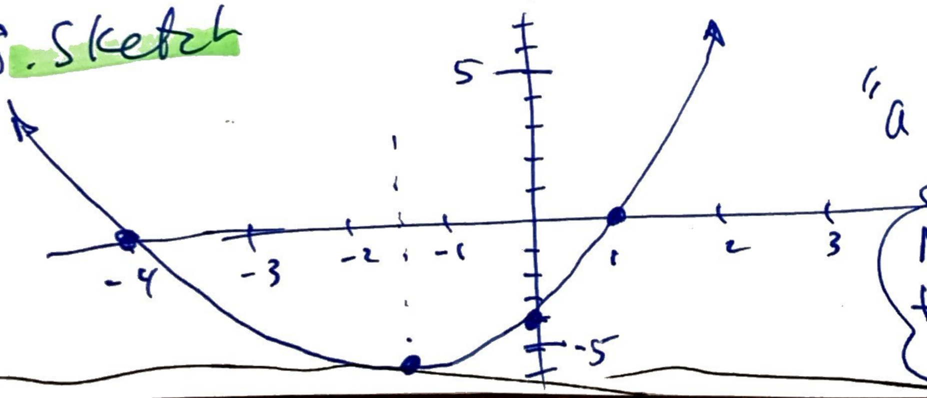
1. Find y-intercept
2. factor Numerators and Denominators
3. Find the zeros : set the numerator to zero.
4. Use $f'(x)=0$ to obtain the critical points.
5. Use number line analysis on $f'(x)$ to locate increasing & Decreasing regions
6. Determine if critical points are max or min (or neither).
7. Use a helper point or two

EX Sketch $f(x)=x^2+3x-4$

1. $f(0)=-4 \rightarrow$ y-int is @ -4, ie $(0,-4)$
3. $f(x)=0 \rightarrow x^2+3x-4=0 \rightarrow (x+4)(x-1)=0 \rightarrow x=-4,1$
2. $f(x)=(x+4)(x-1)$
4. $f'(x)=2x+3 \rightarrow x=-3/2$
5. f' – – – – + + + + + + + + + + + + +, $-3/2$; decreases, increasing



6. $x=-3/2$ is a minimum
7. $f(1)=1^2+3\cdot 1-4 \rightarrow (1,0)$ {we already know this}
   $f(-1)=(-1)^2+3(-1)-4=-6 \quad (-1,-6)$, $f(-3/2)=\left(-\frac{3}{2}\right)^2+3\left(-\frac{3}{2}\right)-4=-\frac{25}{4} \approx -6$
8. Sketch



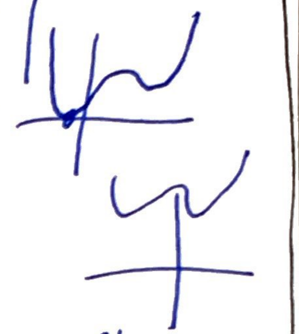
"a parabola"

Note : $f'(x)=0$ is at the line of sym for a parabola

Ex Sketch $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

possibilities

4 zeros　　3-zeros　　2-zeros　　1-zeros　or

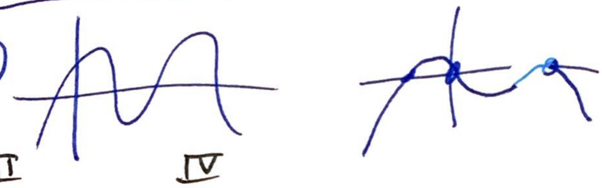


- $a_4 > 0$

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

No zeros

- $a_4 < 0$



. . .

flip all curves upsidown

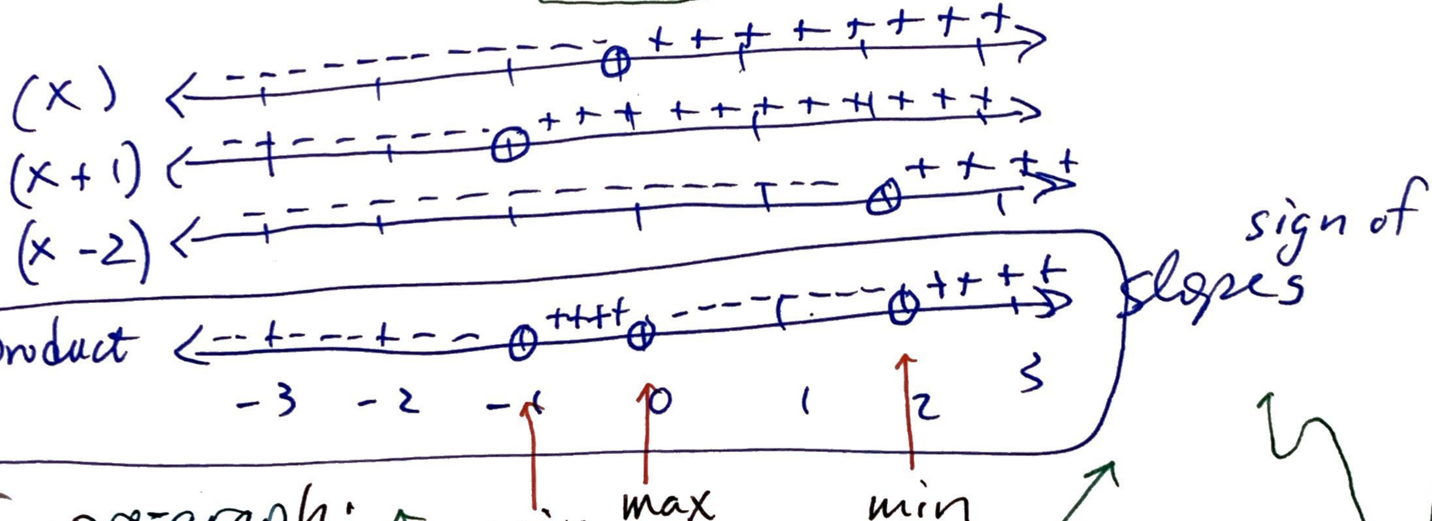
⊗ **Derivative Test:**

$$\boxed{f'(x) = 12x^3 - 12x^2 - 24x}$$

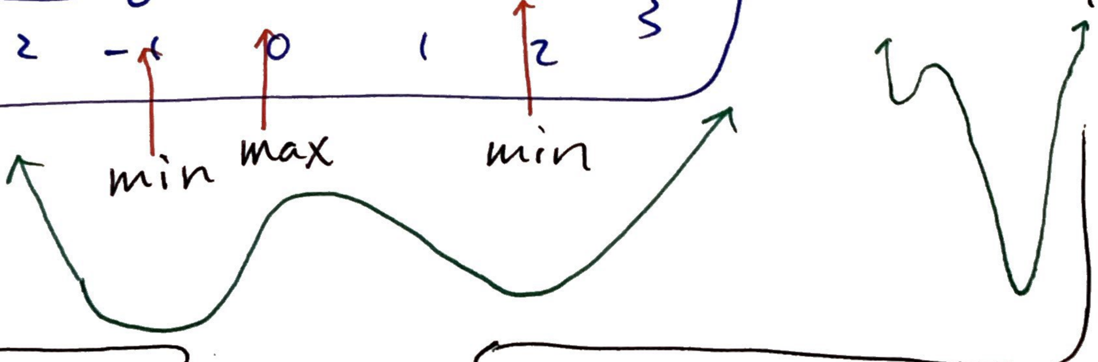
- **sign analysis** - try to factor $f'(x)$:

$$= 12(x)(x^2 - x - 2)$$

$$= 12(x)(x+1)(x-2)$$



- **pregraph:**



- Helper points

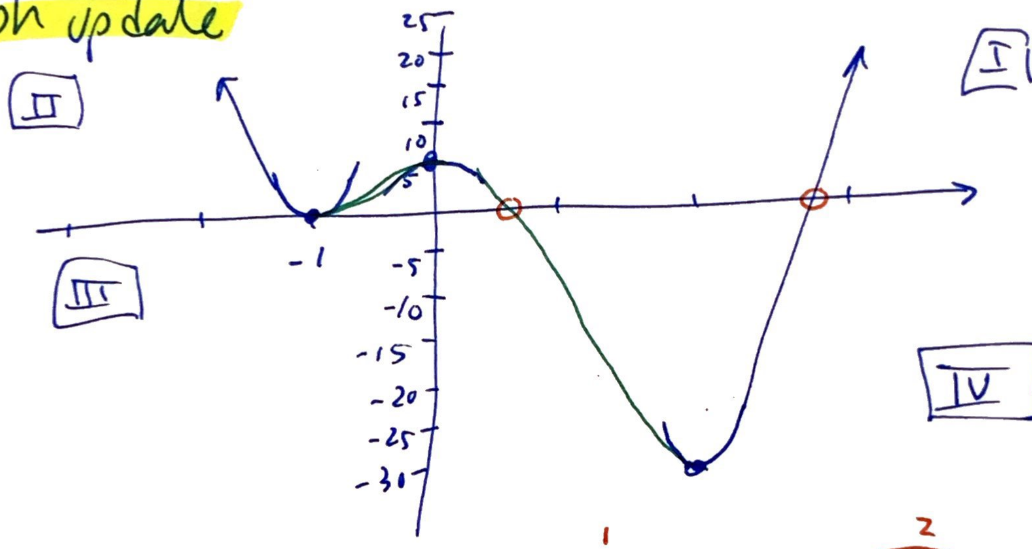
{critical points : −1, 0, 2}

→ $f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5$
$= 3 + 4 - 12 + 5 = 0$ } (−1, 0)

→ $f(0) = 3\cdot 0^4 - 4(0)^3 - 12(0)^2 + 5 = 5$ } (0, 5)

→ $f(2) = 3\cdot 2^4 - 4\cdot 2^3 - 12\cdot 2^2 + 5$
$= 48 - 32 - 48 + 5 = -27$ } (2, −27)

- graph update



- Zeros of $f(x) = +3x^4 - 4x^3 - 12x^2 + 5$

1 2

→ number of zeros : $n = 4$

$\begin{cases} (-1)^{\text{even}} = +1 \\ (-1)^{\text{odd}} = -1 \end{cases}$

→ number of (+) zeros : 2 or 0

$f(-x) = 3(-x)^4 - 4(-x)^3 - 12(-x)^2 + 5$
$= 3x^4 + 4x^3 - 12x^2 + 5$

1

→ number of (−) zeros is : 1

→ rational zero candidates:

$$\boxed{f(x) = 3x^4 - 4x^3 - 12x^2 + 5}$$

$3x^4$: 1, 3 5: 1, 5

candidates: $\pm \frac{1,5}{1,3} \Rightarrow \boxed{\{-5, -\frac{5}{3}, -1, -\frac{1}{3}, \frac{1}{3}, 1, \frac{5}{3}, 5\}}$

synthetic division

	x^4	x^3	x^2	x^1	x^0
-1	3	-4	-12	0	5
		-3	7	5	-5
	3	-7	-5	5	$\boxed{0}$
	x^3	x^2	x^1	x^0	

$$f(x) = (x-(-1))(3x^3 - 7x^2 - 5x + 5)$$
$$\boxed{= (x+1)(3x^3 - 7x^2 - 5x + 5)}$$

use reduced polynomial

Try $x = -1$ again

	x^3	x^2	x^1	x^0
-1	3	-7	-5	5
		-3	10	-5
	3	-10	5	$\boxed{0}$
	x^2	x^1	x^0	

$$\boxed{f(x) = (x+1)^2(3x^2 - 10x + 5)}$$

Just use the quadratic formula:

$$3x^2 - 10x + 5 = 0$$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(5)}}{2 \cdot 3} = \frac{10 \pm \sqrt{40}}{6} = \frac{10}{6} \pm \frac{\sqrt{4 \cdot 10}}{6}$$

6

$$x = \frac{5}{3} \pm \frac{2\sqrt{10}}{6}$$

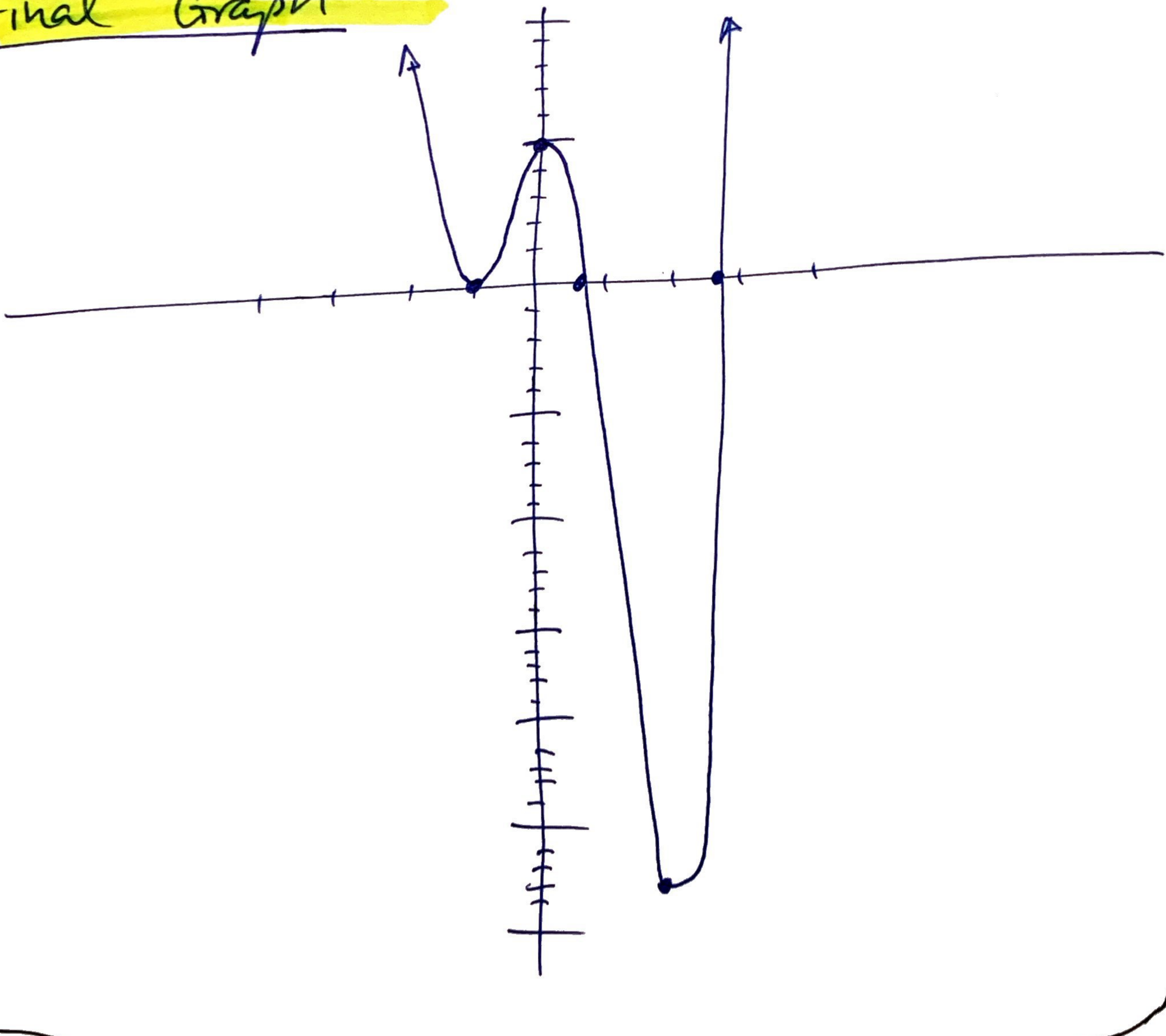
$$x = \frac{5}{3} \pm \frac{\sqrt{10}}{3}$$

$$\boxed{f(x) = (x+1)^2\left(x - \left[\frac{5}{3} - \frac{\sqrt{10}}{3}\right]\right)\left(x - \left[\frac{5}{3} + \frac{\sqrt{10}}{3}\right]\right)}$$

zeroes @ $x = -1, -1, \underbrace{\frac{5}{3} - \frac{\sqrt{10}}{3}}_{0.7}, \frac{5}{3} + \frac{\sqrt{10}}{3}$

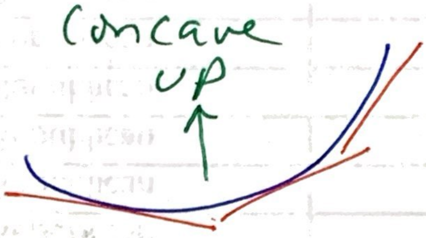
0.613   2.72

- Final Graph

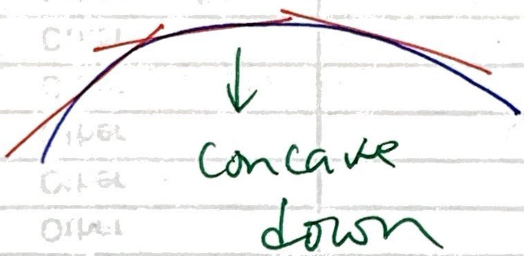


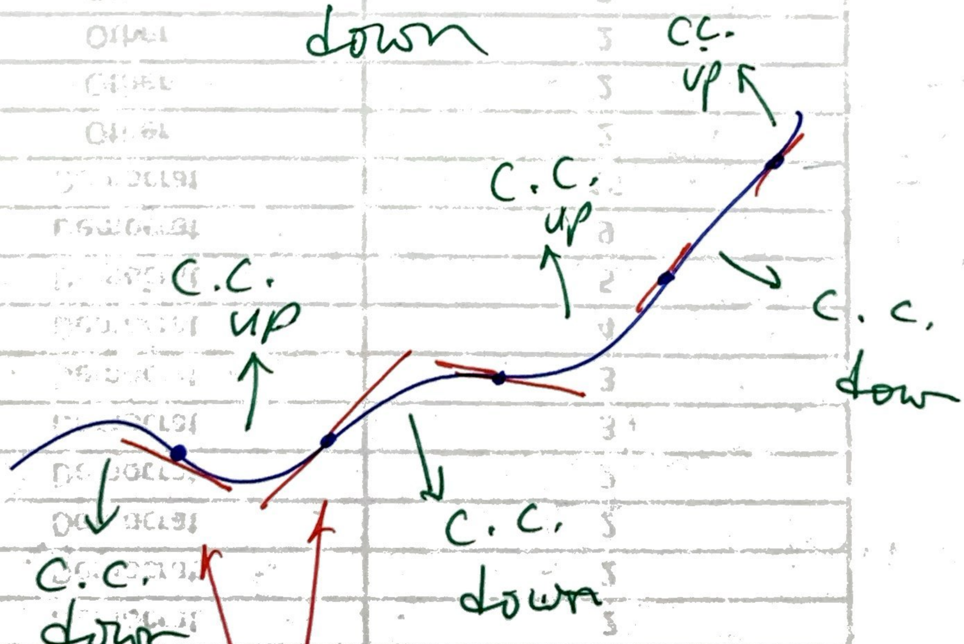
III The second derivative, f'', and shape of f

Def: a curve has a concave upward portion when the curve "cups" upwards



Def: a curve has a concave down portion when it "cups" down





Def: Inflection points.

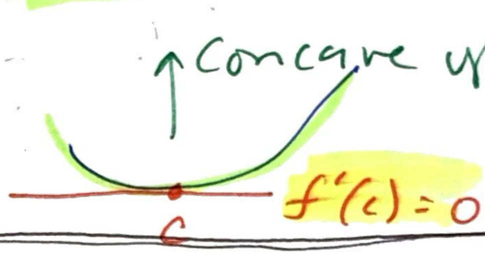
$f(x)$ changes concavity @ I.P.

⊛ The second derivative test.

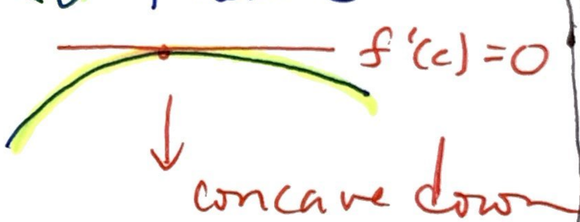
- Suppose $f''(x)$ is continuous at $x=c$

Notation
$$\left\{ f'(x) \equiv \frac{df(x)}{dx}, \quad f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^2 f}{dx^2} \right\}$$

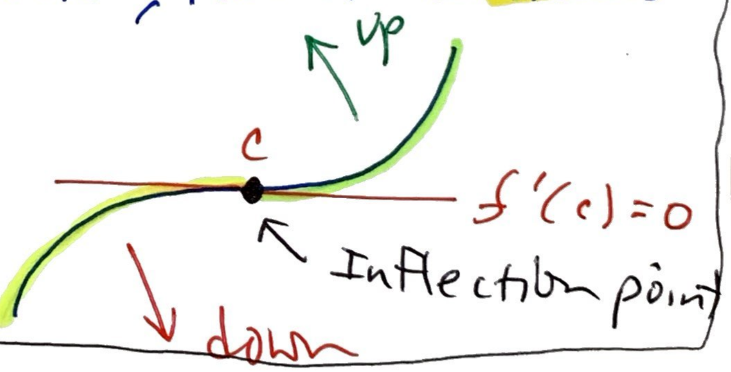
(a) If $f'(c)=0$ and $f''(c)>0$ then "c" is a local minimum



(b) If $f'(c)=0$ but $f''(c)<0$ then "c" is a local maximum



(c) If $f'(c)=0$ and $f''(c)=0$ then "c" is neither a max nor min, rather a horizontal inflection point

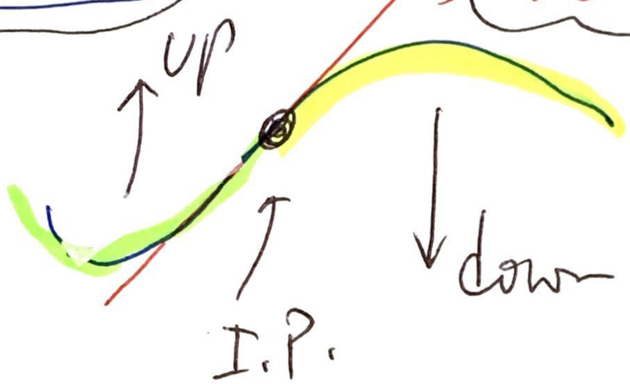


⊛ Note 1: when $f''(x)=0$ then x may be an inflection point, but need not be.
ex: straight line

* **Note 2:** If $f''(c) = 0$ we have an inflection point but $f'(c)$ need not be zero.

Ex

$f' \neq 0$ but $f'' = 0$



Ex Consider $f(x) = x^3 - x^2 + 2x - 11$

(a) what are the critical points?

$$f'(x) = 3x^2 - 2x + 2$$

$$0 = 3x^2 - 2x + 2 = (x \quad)(x \quad)$$

trying to factor

3, 1   2, 1

innies $(1 \cdot 2)$

outies $(3 \cdot 1)$

Q: does $2 \pm 3 \overset{?}{=} -2$

A: No

3, 1   2, 1

lefties

righties

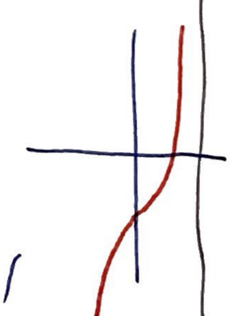
Q: $(3 \cdot 2 = 6) \pm (1 \cdot 1) \overset{?}{=} -2$

A:

Quad. Formula:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(2)}}{2 \cdot 3}$$

$$= \frac{2 \pm \sqrt{-20}}{6}$$ No Real zeros!



Conclusion: No critical points

- Inflection points:

$$f'(x) = 3x^2 - 2x + 2$$

So $f''(x) = 6x - 2$

$$0 = 6x - 2 \longrightarrow \boxed{x = 1/3}$$

and $f(\frac{1}{3}) = (\frac{1}{3})^3 - (\frac{1}{3})^2 + 2(\frac{1}{3}) - 11$

$$= \frac{1}{27} - \frac{1}{9} + \frac{2}{3} - 11$$

$$= \frac{1}{27} - \frac{3}{27} + \frac{18}{27} - \frac{11 \cdot 27}{27}$$

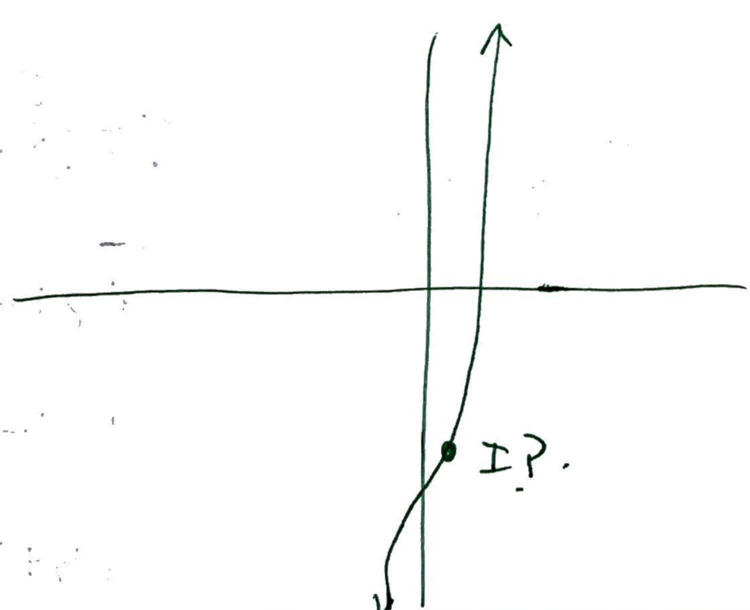
$$= \frac{1 - 3 + 18 - 297}{27}$$

$$= -281/27 \approx -10.41$$

$$\begin{array}{r} 27 \\ \times 11 \\ \hline 270 \\ + 27 \\ \hline 297 \\ +3 \\ 300 \\ -19 \\ \hline 281 \end{array}$$

$IP = \boxed{\left(\frac{1}{3}, -\frac{281}{27}\right)} \approx (0.33, -10.41)$

- sketch



Ex $f(x) = -2x^3 + 3x^2 + 36x$

Not formally following the steps...

(i) $f'(x) = -6x^2 + 6x + 36$

$0 = -6x^2 + 6x + 36 \quad \div -6$

$0 = x^2 - x - 6$

2, 3

$0 = (x-3)(x+2)$ $\quad \boxed{x = 3, -2}$

(ii) $f''(x) = -12x + 6$

Recall

max

$f'' < 0$

min

$f''(x) > 0$

$f''(3) = -12 \cdot 3 + 6 = -18 + 6 < 0$ concaved down $\Rightarrow \boxed{\text{max}}$ $x = 3$

$f''(-2) = -12(-2) + 6$

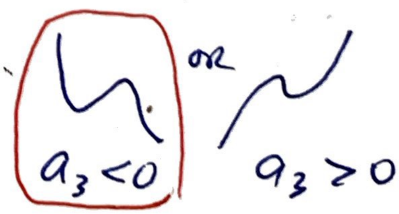
$= 24 + 6 > 0$ $\quad \boxed{x = -2}$ is min

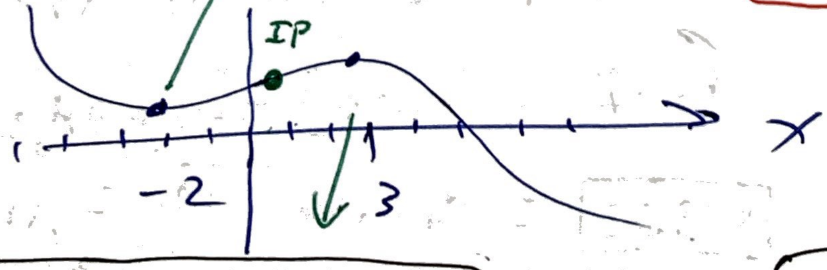
(iii) I. Point. $f''(x) = 0$

$f''(x) = -12x + 6$

$0 = -12x + 6$ $\quad \boxed{x = 1/2}$

Pre-sketch! cubic





(iv) $f(-2) = -2(-2)^3 + 3(-2)^2 + 36(-2) = -44$

$f(1/2) = -2(\frac{1}{2})^3 + 3(\frac{1}{2})^2 + 36(\frac{1}{2}) = 18.5$

$f(3) = -2(3)^3 + 3(3)^2 + 36(3) = 81$

helper pts

$f(0) = -2(0)^3 + 3(0)^2 + 36(0) = 0$

$f(-3) = -2(-3)^3 + 3(-3)^2 + 36(-3) = -27$

$f(4) = -2(4) + 3(4)^2 + 36(4) = 40$

(v) zeros of $f(x)$:

$f(x) = -2x^3 + 3x^2 + 36x$

$0 = -2x^3 + 3x^2 + 36x$

$0 = -x(2x^2 - 3x - 36)$

6,6
9,~~4~~

$0 =$

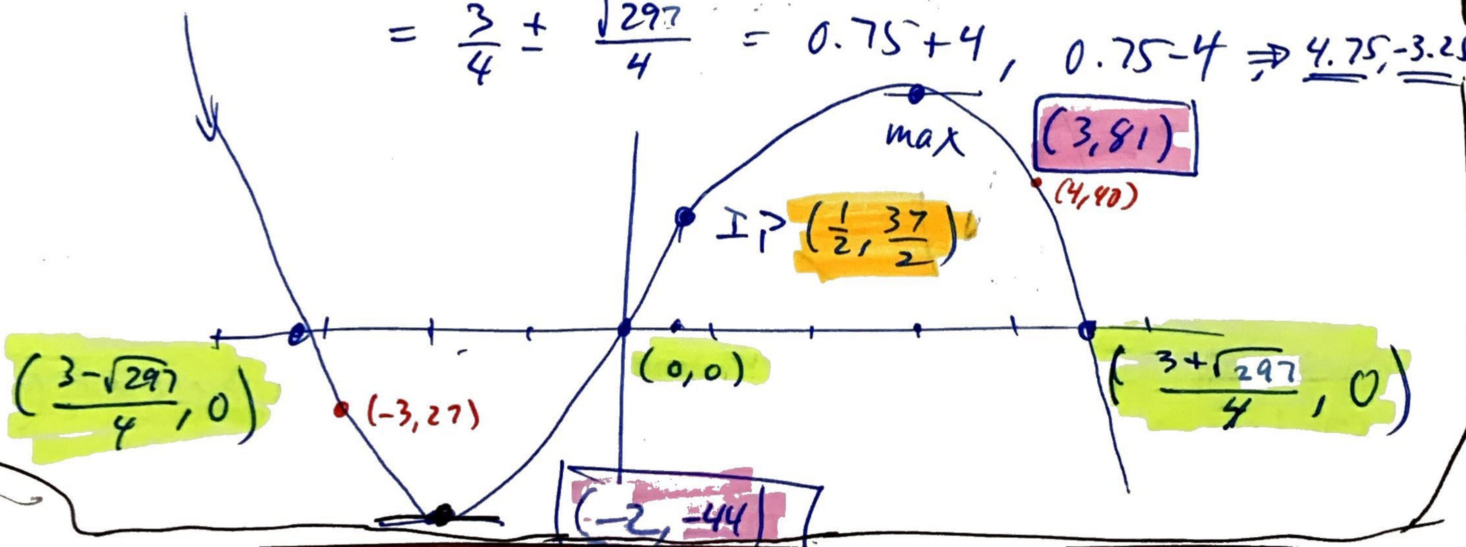
2,1

9,4
6,6
18,2
36,1
12,3

72
4

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-36)}}{2 \cdot 2} = \frac{3 \pm \sqrt{9 + 288}}{4}$

$= \frac{3}{4} \pm \frac{\sqrt{297}}{4} = 0.75 + 4,\ 0.75 - 4 \Rightarrow 4.75, -3.25$



EX Sketch $f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$

Not formally following the steps:

Alt. form $f(x) = \dfrac{x^2+x-1}{x^2}$

(a) Vertical Asymptotes: (Denom = 0) @ $\boxed{x=0}$ V

(b) Horizontal Asymptote: $\dfrac{\text{deg on top}}{\text{deg on bot.}}$

then $y_{\text{Hor. Asympt.}} = \frac{1}{1} = 1$ $\boxed{y=1}$ H.A.

(c) Increase/decrease:

$$f'(x) = \frac{-1}{x^2} + \frac{2}{x^3} = \frac{-1}{x^3}(x-2)$$

line analysis

x^3: − − − − (0) + + + + +

$x-2$: − − − − − − (2) + + +

so $\dfrac{x-2}{x^3}$: + + + + + (0) − − − − (2) + + +

but $-\dfrac{x-2}{x^3} =$ − − − − (0) + + + (2) − − −

decreasing (0) increasing (2) decreas.

(d) Concavity

$$f'' = \left(-\frac{1}{x^2} + \frac{2}{x^3}\right)'$$

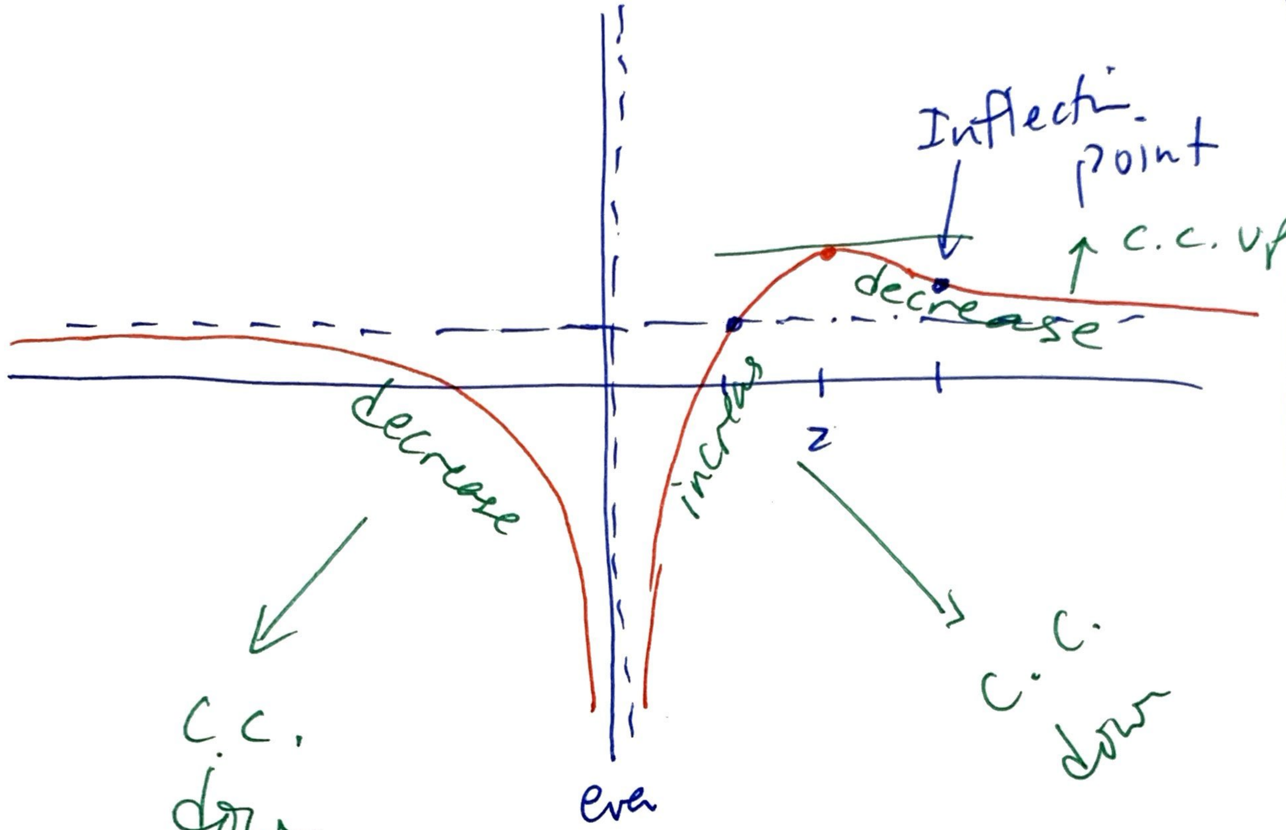
$$= \frac{2}{x^3} - \frac{6}{x^4}$$

$$= \frac{2}{x^4}(x-3)$$

I.P. at 3

concave down (3) c.c. up

(e) assemble sketch



• Crossing HA? $\underbrace{f(x)}_{}$ = 1 set to 1 and solve:

so $\dfrac{x^2+x-1}{x^2} = 1$

solve $\cancel{x^2}+x-1=\cancel{x^2}$

results: $\boxed{x=1}$ is the only crossing of the HA $y=1$

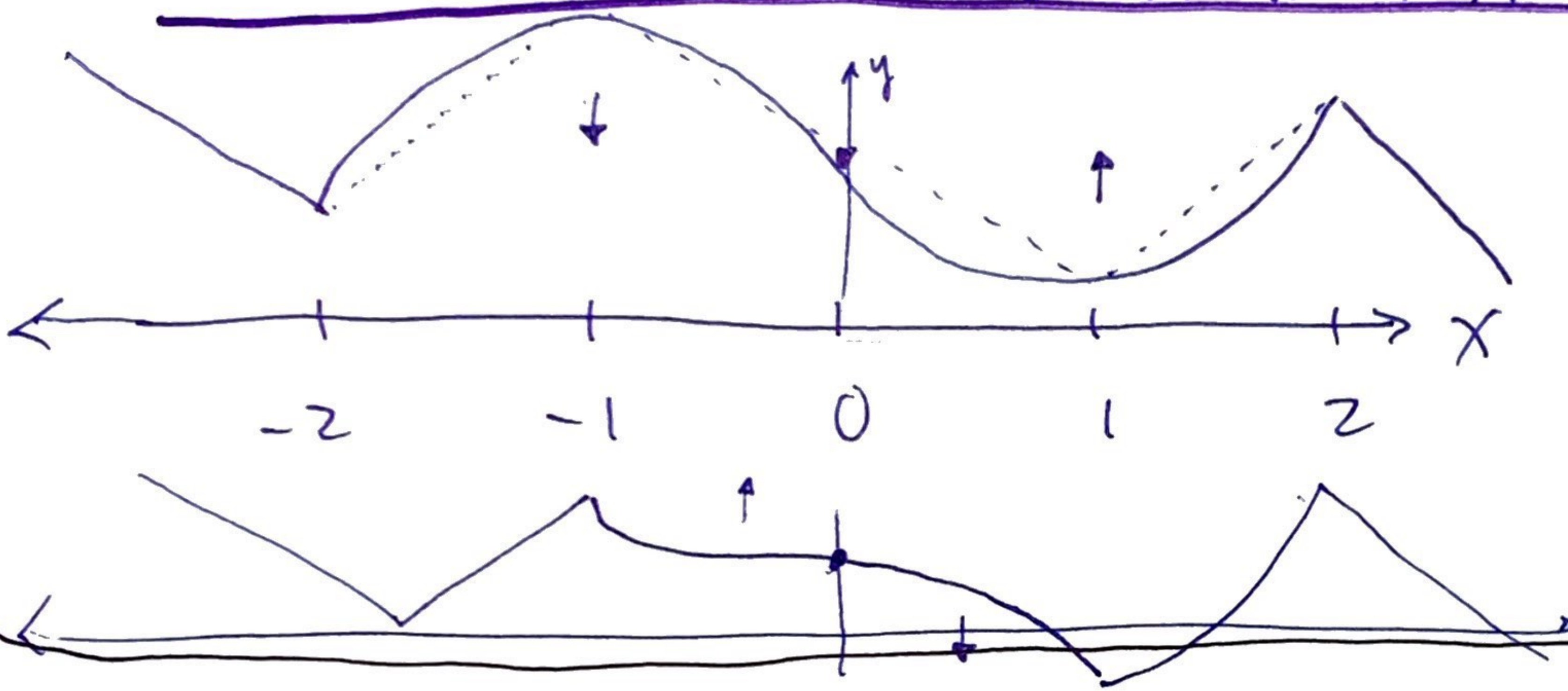
Ex sketch a possible graph of a continuous function that satisfies

I. $f' < 0$ in $x \in (-1, 1)$ } i.e. $|x| < 1$

II. $f' > 0$ in $\underbrace{1 < x < 2 \text{ and } -2 < x < -1}_{\text{i.e. } 1 < |x| < 2}$

III. $f' = -1$ for $x > 2$ and for $x < -2$ } i.e. $|x| > 2$

IV: f has an inflection point @ $(0, 1)$



two possible graphs.