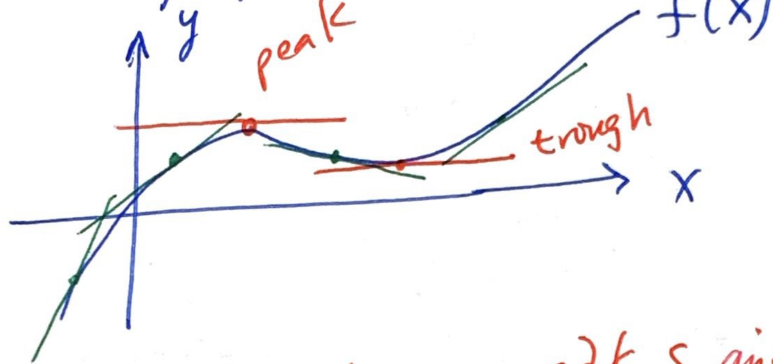
Chapter 3 Applications of Differentiation

3.1 Extrema and rational zeros and synthetic division review

- Consider the graph of a function $f(x)$



Q: How do we find the peaks and troughs?

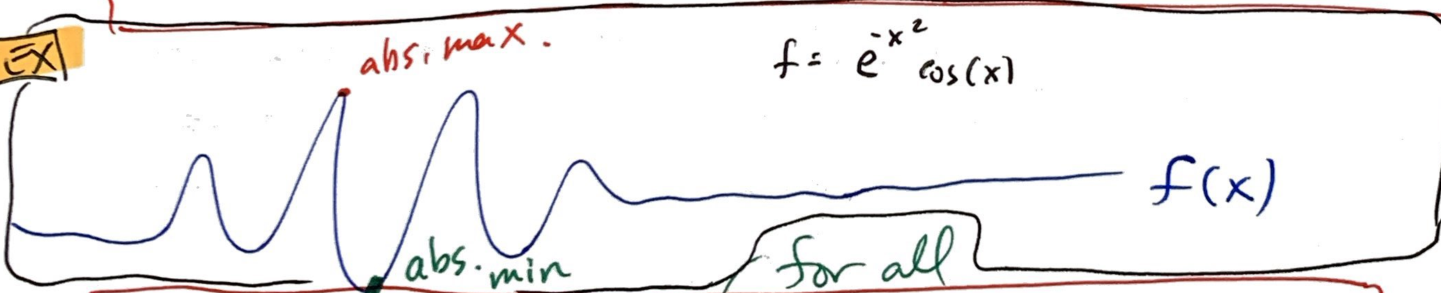
A: $\boxed{\text{the tangent line has zero slope}}$

Def: Extrema are values of the function that are minimums or maximums

DEF: Let "c" be a number in the Domain of $f(x)$.

- Then $f(c)$ is an **absolute maximum** value of $f(x)$ if $f(c) \geq f(x)$ for all x in D_f

EX



- $f(c)$ is an **absolute minimum** of $f(x)$ if $f(c) \leq f(x)\ \forall\ x \in D_f$

for all

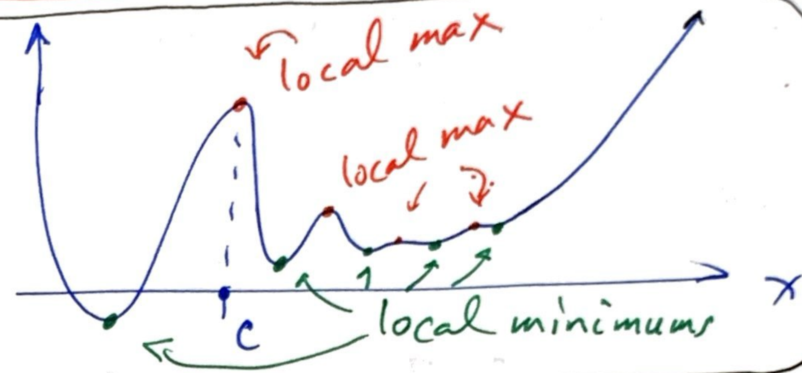
Domain of $f(x)$

element of

DEF: $f(c)$ is a

- **local maximum** of $f(x)$ if $f(c) \geq f(x)$ when x is near "c"

Ex



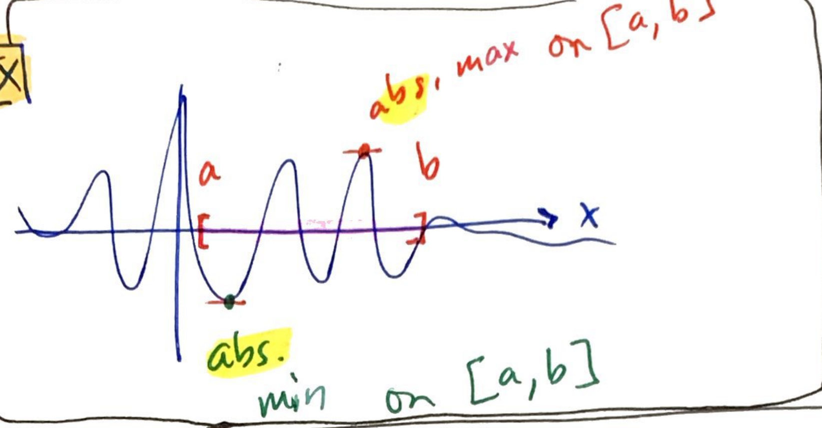
- $f(c)$ is a **local min** of $f(x)$ if $f(c) \leq f(x)$ when x is near "c"

THM — **The Extreme Value Theorem** (EVT)

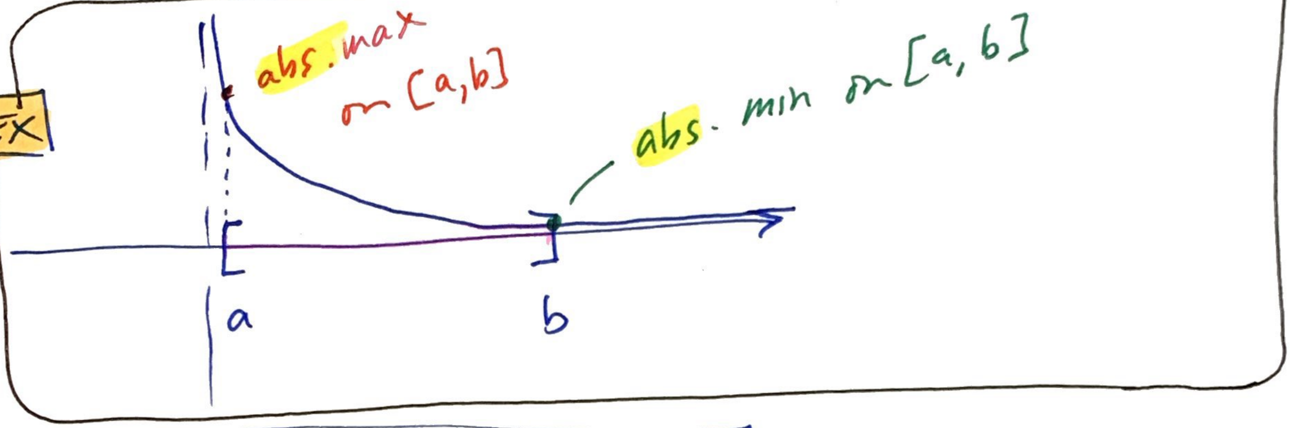
If $f(x)$ is **continuous** on closed intvl $[a, b]$,
then $f(x)$ attains an **absolute max** value, $f(c)$, and and **absolute minimum**, $f(d)$ inside or at the boundaries of $[a, b]$

(*) This thm tells us that these extrema exists but does not tell us where in D_f they are located. (*)

EX



EX



THM — Fermat's Theorem

If $f(x)$ has a local extrema at $x = c$ and
if $f'(c)$ exists
Then $f'(c) = 0$

logic Bubble

- $a \to b$ read "a" implies "b" is a conditional

  if "a" then "b"

- $b \to a$ is the converse of $a \to b$
- $\sim b \to \sim a$ is read as "not "b" implies not "a""
  and is called the contrapositive of $a \to b$

Generally speaking, $a \to b$ does not work backwards.

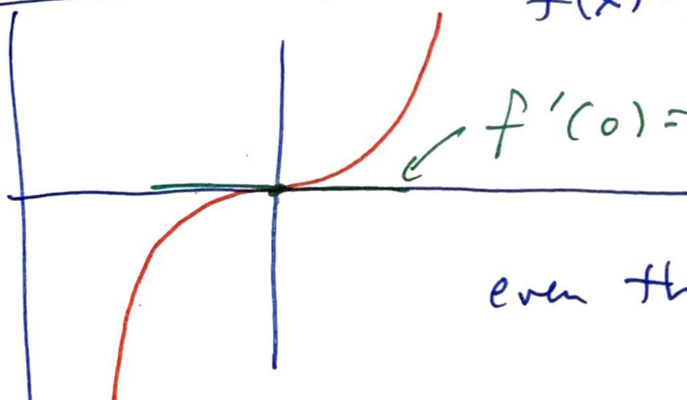
Ex: If you water the lawn then it will be green. $a \to b$

Ex: If the lawn is green then you watered the lawn. $b \to a$

So if $f'(x) = 0$ @ $x = c$ we cannot guarantee an extrema exists at "c"

Counter example

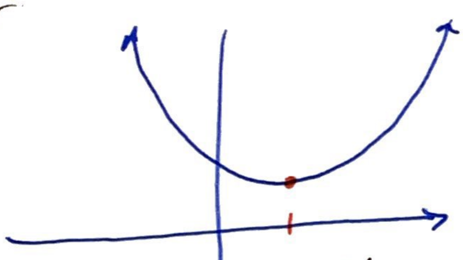
$f(x) = x^3$



$f'(0) = 0$ yet $x = 0$ is NOT an extrema.

even though $f'(0)$ exists.

EX Where is $f(x) = \overset{a}{3x^2} \overset{b}{-x} \overset{c}{+1}$ at it's minimum?



$c = 1/6$

$\left\{ x_s = -\frac{(-1)}{2 \cdot 3} = 1/6 \right\}$

$f'(x) = 6x - 1$ will be zero.

So $0 = 6x - 1$

$1 = 6x$

$\boxed{1/6 = x}$ ← location of the minimum

- The value of the minimum is

$$f\left(\tfrac{1}{6}\right) = 3\left(\tfrac{1}{6}\right)^2 - \left(\tfrac{1}{6}\right) + 1$$

$$= \frac{3}{36} - \frac{6}{36} + \frac{36}{36}$$

$$= \frac{33}{36} \text{ or } \boxed{11/12}$$

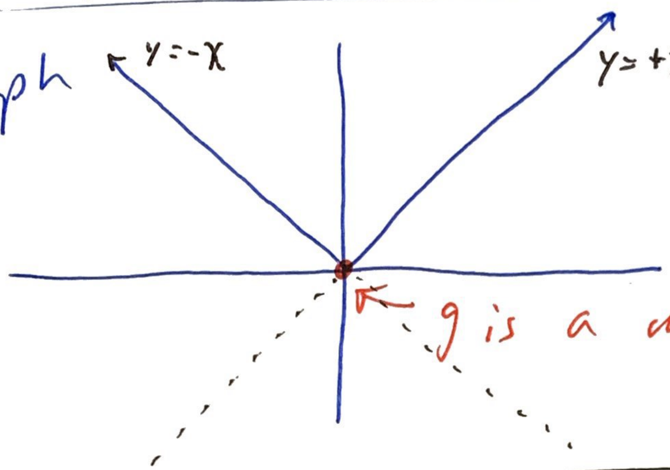
minimum of $f(x)$ which occurs at $x = 1/6$

BTW: Alg II/Pre-Calc we found the symmetry line for parabolas was $x = \frac{-b \pm \sqrt{b^2 - 4a}}{2a}$, $f = ax^2 + bx + c$

- another counterexample

$\boxed{\text{EX}}$ Consider $g(x) = |x|$. Is $g'(x) = 0$ when $g(x)$ is a minimum?

graph

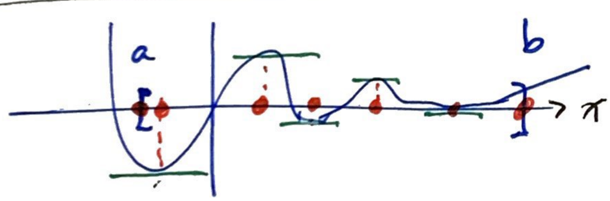


$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

g is a min here (actually not just a min, an absolute minimum)

Def: the values in the D_f where $f'(x) = 0$ or where x is a boundary, are called the critical values of $f(x)$

$\boxed{\text{EX}}$



{ critical points are the values of "x" not the values of $f(x)$.

⊛ procedures for locating the extrema of $f(x)$ on $[a, b]$

(i) Find all critical values of $f(x)$
(ii) Find all of the values of $f(x)$ at the crit. values
(iii) The max value of "f" is the abs. max
The min value of "f" is the abs. min on the interval $[a, b]$

Ex Find the absolute maximum and minimum of the function $f(x) = \frac{x}{x^2 - x + 1}$ on $[0,3]$

(i) critical points:

- $f'(x) = \frac{(x)'(x^2-x+1) - (x)(x^2-x+1)'}{(x^2-x+1)^2}$

$$0 = \frac{1\cdot(x^2-x+1) - x(2x-1)}{(\quad)^2}$$

$$0 = \frac{-x^2+1}{(x^2-x+1)}$$

→ examine numerator only

$$0 = -x^2 + 1$$

so $x^2 = 1$

so $x = +1, -1$ ← not inside interval

Q: is $(x^2 - x + 1) = 0$ on $[0,3]$?
a b c

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2\cdot 1}$$

complex since $(-1)^2 - 4\cdot1\cdot1 < 0$

A: No

(ii) find values of $f(x)$ at critical points → $\boxed{x = 0, 1, 3}$:

- $f(0) = \frac{0}{0^2 - 0 + 1} = 0$
- $f(1) = \frac{1}{1^2 - 1 + 1} = 1$
- $f(3) = \frac{3}{3^2 - 3 + 1} = \frac{3}{7}$

(iii) assess

value of $f(x)$ location

- min of $f(x)$ on $[0,3]$ is $\boxed{0}$ @ $x = 0$
- max of $f(x)$ on $[0,3]$ is $\boxed{1}$ @ $x = 1$

Ex Determine all critical point locations:

$$R(w) = \frac{w^2+1}{w^2-w-6}$$

$$R'(w) : \frac{d}{dw}\left(\frac{w^2+1}{w^2-w-6}\right)$$

$$R' = \frac{(w^2+1)'(w^2-w-6) - (w^2+1)(w^2-w-6)'}{(w^2-w-6)^2}$$

$$R' = \frac{(2w)(w^2-w-6) - (w^2+1)(2w-1)}{(\quad)^2}$$

$$R' = \frac{(2w^3-2w^2-12w) - [2w^3-w^2+2w-1]}{(\quad)^2}$$

$$R' = \frac{\cancel{2w^3} - 2w^2 - 12w - \cancel{2w^3} + w^2 - 2w + 1}{(\quad)^2}$$

$$R' = \frac{-w^2-14w+1}{(\quad)^2}$$

$$\boxed{R' = -\frac{w^2+14w-1}{(w^2-w-6)^2}}$$

Denominator factors to $(w-3)(w+2)$ and has zeros at $3, -2$

Set $R' = 0$ which means only the numerator is set to zero.

$$\boxed{0 = w^2 + 14w - 1} \qquad (w\ \ -1)(w\ \ 1)$$

- won't factor so use quadratic formula.

$$w = \frac{-14 \pm \sqrt{14^2 - 4(1)(-1)}}{2 \cdot 1}$$

$$\begin{array}{r} 14 \\ 10 \\ \hline 140 \end{array}$$

$$= \frac{-14 \pm \sqrt{196 + 4}}{2}$$

$$\begin{array}{r} 14 \\ 4 \\ \hline 56 \end{array}$$

$$= \frac{-14 \pm \sqrt{200}}{2}$$

$$\begin{array}{r} 140 \\ 56 \\ \hline 196 \end{array}$$

$$= \frac{-14 \pm \sqrt{2 \cdot 100}}{2}$$

$$= \frac{-14 \pm \sqrt{2}\sqrt{100}}{2}$$

$$= \frac{-14 \pm 10\sqrt{2}}{2} = -\frac{\cancel{14}^{\,7}}{\cancel{2}} \pm \frac{\cancel{10}^{\,5}}{\cancel{2}} \cdot \sqrt{2}$$

$$\boxed{w = -7 \pm 5\sqrt{2}}$$

(neither of these collide with Denom. zeros of $-2, 3$)

- critical points locations at $w =$

$$\boxed{\{-7 - 5\sqrt{2}\ ,\ -7 + 5\sqrt{2}\}}$$

Ex Find the critical points of $y=3x^3-4x^2+x-9$ on $[-4,3]$

(i) critical points:

$y'=0$: $\boxed{y' = 9x^2-8x+1}$

• try factoring: $0 = 9x^2-8x+1$

9, 1   1, 1

3, 3

$0 = (9x-1)(x-1)$

-1

-9

-10

• use quadratic:

$$x = \frac{-(-8)\pm\sqrt{(-8)^2-4(9)(1)}}{2\cdot 9}$$

$$x = \frac{8\pm\sqrt{64-36}}{18}$$

$$x = \frac{8\pm\sqrt{28}}{18}$$

4.7

$$x = \frac{8\pm 2\sqrt{7}}{18}$$

$\frac{\sqrt{7}}{9} \approx \frac{2.7\text{-ish}}{9}$

$$\boxed{x = \frac{4}{9}\pm\frac{\sqrt{7}}{9}}$$

$\approx 0.5 \pm 0.3$ 0.8   0.2

Interval $[-4,3]$

critical points: $\boxed{\left\{-4, \boxed{\frac{4}{9}-\frac{\sqrt{7}}{9}}, \boxed{\frac{4}{9}+\frac{\sqrt{7}}{9}}, 3\right\}}$

stop here

* Synthetic Division Review

Given a polynomial $2x^3 - 9x^2 + 7x + 6 = 0$

if 2 is a zero then when we synthetically divide 2 into the polynomial we will have a remainder of zero.

$$
\begin{array}{r|rrrr}
2 & 2 & -9 & 7 & 6 \\
 & & 4 & -10 & -6 \\
\hline
 & 2 & -5 & -3 & \boxed{0}
\end{array}
$$

remainder!

reduced polynomial

- So

$2x^3 - 9x^2 + 7x + 6 = 0$ factors to become

$(x-2)(2x^2 - 5x - 3) = 0$

- try $\underline{\underline{x=1}}$ as a root; but now use the reduced polynomial since we found a root already

$$
\begin{array}{r|rrr}
1 & 2 & -5 & -3 \\
 & & 2 & -3 \\
\hline
 & 2 & -3 & -6
\end{array}
\neq 0
$$

* $x = +1$ is not a zero

$$
\begin{array}{r|rrr}
-1 & 2 & -5 & -3 \\
 & & -2 & 7 \\
\hline
 & 2 & -7 & 4
\end{array}
\neq 0
$$

* $x = -1$ is not a zero

Review

Factor : $0 = 2x^3 - 9x^2 + 7x + 6$

Thm if a zero to a polynomial has a rational zero, that zero is one factor of the constant term divided by a factor of the leading coefficient.

Ex

$$2x^3 - 9x^2 + 7x + 6 = 0$$

2, 1   1, 2, 3, 6

• Thm: $\pm \dfrac{1,2,3,6}{2,1} = \pm\left\{ \dfrac{1}{2}, \underbrace{\dfrac{2}{2}}_{=1}, \dfrac{3}{2}, \dfrac{6}{2}, \dfrac{1}{1}, \dfrac{2}{1}, \underbrace{\dfrac{3}{1}}_{=1}, \dfrac{6}{1} \right\}$

$$= \boxed{\pm\left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 6 \right\}}$$

List all rational zero candidates from the most negative to the most positive ...

$$\boxed{\left\{ -6, -3, -2, -\frac{3}{2}, -1, -\frac{1}{2}, +\frac{1}{2}, 1, \frac{3}{2}, 2, 3, 6 \right\}}$$

Next we syntheically divide these into the polynomial to find zeros... if any rational number zeros even exist!

Ex Find critical points:

$$y(x) = \frac{x^4}{2} - 3x^3 + \frac{7}{2}x^2 + 6x - 11$$

- Diff't:

$$y' = 4\cdot\frac{x^3}{2} - 3\cdot 3x^2 + \frac{7}{2}\cdot 2x + 6$$

$$\boxed{y' = 2x^3 - 9x^2 + 7x + 6}$$

- Set to zero:

$$0 = 2x^3 - 9x^2 + 7x + 6$$

- try factor by grouping:

$$0 = (2x^3 - 9x^2) + (7x + 6)$$

$$0 = x^2(2x-9) + (7x+6) \quad \times$$

$$0 = (2x^3 + 7x) - (9x^2 - 6)$$

$$= x(2x^2+7) - 3(3x^2-2) \quad \times$$

Factor by grouping fails

- Next we try rational roots.

$$\pm\frac{\text{factors of } 6}{\text{factors of } 2} = \frac{\{1,2,3,6\}}{\{1,2\}} \Longrightarrow$$

- we can try all the roots in the list of rational number candidates (previous example)

$$\{-6, -3, -2, -\tfrac{3}{2}, \cancel{-1}, \tfrac{-1}{2}, \tfrac{1}{2}, \cancel{1}, \tfrac{3}{2}, \checkmark 2, 3, 6\}$$

- we tried $x=2$ in the previous example and it worked!
- try $x=3$

$$\begin{array}{r|rrr} & x^2 & x^1 & x^0 \\ 3 & 2 & -5 & -3 \\ & & 6 & 3 \\ \hline & 2 & 1 & \boxed{0} \\ & x^1 & x^0 & \end{array}$$

$$(2x-5x-3) = (x-3)(2x+1)$$

- So

$$2x^3-9x^2-7x+6 = 0$$ factors into

$$\boxed{(x-2)(x-3)(2x+1) = 0}$$

$\Rightarrow$ $\underline{x = 2, 3, -\tfrac{1}{2}}$ zeros of the polynomial $y' = 2x^3-9x^2+7x+6$