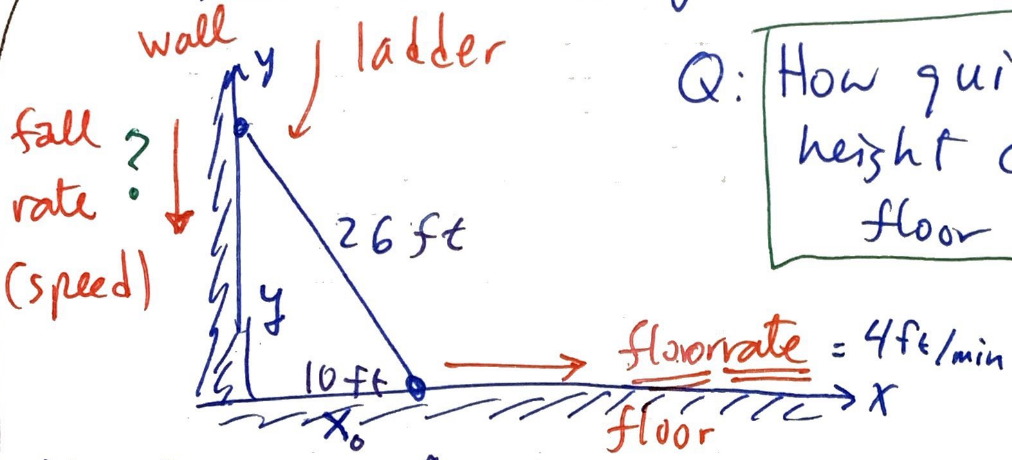
2.8 (Word Problems) Related Rates

- In this section we examine how variables change w.r.t. to time, or even just each other.

Ex Ladder Slipping Down the Wall



Q: How quickly does the wall height change as the floor rate is fixed at 4 ft/min ?

(i) Diagram ↑ (ii) Variables $l_0 = 26$ ft , $x_0 = 10$ ft

(iii) eqns: let x = the ladder's floor displacement from the wall
let y = the ladder's hieght from the floor

then Pythagores's Thm says $\boxed{x^2 + y^2 = 26^2}$

(iv) Do ~~the~~ math:

we want $\dfrac{dy}{dt} = ?$, we know $\dfrac{dx}{dt} = 4$ ft/min

- Diff't $x^2 + y^2 = 26^2$ w.r.t. time to get:

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt} 26^2$$

— or —

2

$$\frac{dx^2}{dt} + \frac{dy^2}{dt} = 0$$

power rule with chain rule.

$$\boxed{2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0}$$

- Solve for $\frac{dy}{dt}$:

$$\cancel{2}y\frac{dy}{dt} = -\cancel{2}x\frac{dx}{dt} \qquad \div y$$

$$\boxed{\frac{dy}{dt} = -\frac{x}{y}\left(\frac{dx}{dt}\right)}$$

This relates floor speed with wall speed.

speed on floor

speed down wall

- Lets Apply This

$\boxed{@\ t=0}$ $x_0 = 10\text{ ft}$, $y_0 = \sqrt{26^2-10^2} = \underline{\underline{24\text{ft}}}$

"evaluate at $t=0$"

$$\Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = -\frac{x_0}{y_0}\left.\left(\frac{dx}{dt}\right)\right|_{t=0}$$

$$\left.\frac{dy}{dt}\right)_{t=0} = -\frac{(\cancel{10}^{\,5}\text{ft})}{\cancel{24}\text{ft}\ \cancel{6}\ 3}\cdot\cancel{4}\text{ft}/\text{min}$$

$$= \boxed{-\frac{5}{3}\text{ ft}/\text{min}} = -1\frac{2}{3} = -1.6\overline{6} \approx -1.67 \text{ ft}/\text{min}$$

down the wall

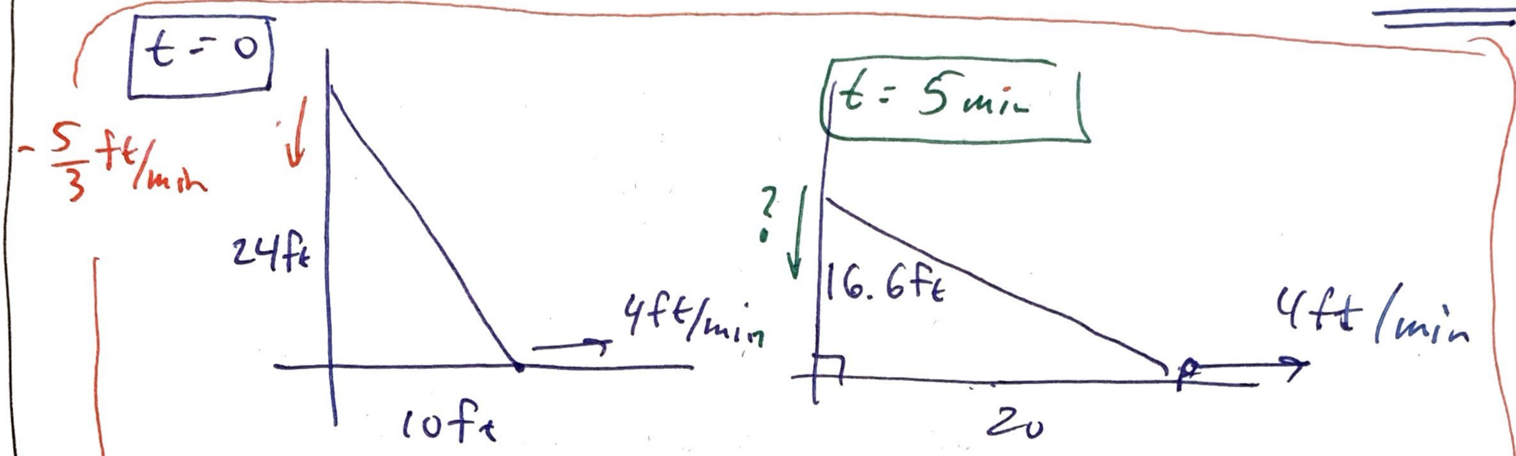
• What is the speed 5 minutes later?

③

$\boxed{@\ t_1 = 5\ min}$, $X_1 = 4\frac{ft}{\cancel{min}} \cdot 5\,\cancel{min} = \underline{\underline{20\,ft}}$

$Y_1 = \sqrt{26^2 - 20^2} = \sqrt{276} = \underline{\underline{2\sqrt{69}}}$

$\approx \underline{\underline{16.6\,ft}}$



$X_1 = 20\,ft$, $y_1 = 16.6\,ft$, $\frac{dx}{dt} = 4\,ft/min$

• Use rate relation

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

• plug in data...

$$\Longrightarrow \frac{dy}{dt} = -\left(\frac{20\cancel{ft}}{16.6\cancel{ft}}\right)\left(4\,ft/min\right)$$

$$= \boxed{-4.82\ ft/min} \ @\ t = 5\ min$$

vs.

-1.67 ft/min @ $t = 0$ minutes

speeding up ... get out of the way!!

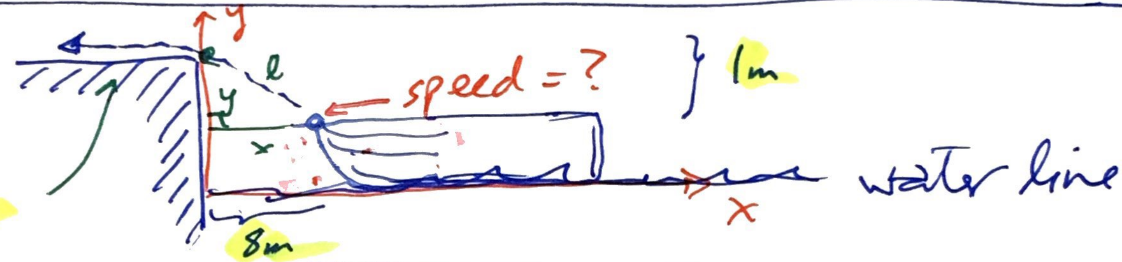
EX Reeling in a boat

A boat is being pulled into a dock by a rope. The front top surface of the boat is 1 m below the dock.

Q: If the rope is reeled in at 1 m/s, how fast does the boat approach the dock at 8 m out.

(i) Diagram

given $\frac{d\ell}{dt} = 1\,m/s$



pythag'n formula: $\boxed{\ell^2 = x^2 + y^2}$

speed of surface of boat $= \frac{dx}{dt}$ = we seek

(ii) variables $\boxed{y_0 = 1m,\ x_0 = 8m,\ \frac{d\ell}{dt} = 1\,m/s,\ \frac{dx}{dt} = ?}$

(iii) eqn $x^2 + y^2 = \ell^2$ ← not fixed this time

(iv) Solve $\boxed{2x\frac{dx}{dt} + 2y\cancel{\frac{dy}{dt}}^{\,0} = 2\ell\frac{d\ell}{dt}}$

$$\cancel{2}x\frac{dx}{dt} = \cancel{2}\ell\frac{d\ell}{dt}$$

$$\boxed{\frac{dx}{dt} = \frac{\ell}{x}\left(\frac{d\ell}{dt}\right)}$$

related rates of rope vs Boat speed

• at $t=0$ $\quad l_0 = \sqrt{8^2+1^2} = \sqrt{65} \approx 8m$

$x_0 = 8m$

So plug data in: $\left.\dfrac{dx}{dt}\right|_{t=0} = \dfrac{l_0}{x_0}\left.\dfrac{dl}{dt}\right)_{t=0}$

$$= \frac{\sqrt{65}\,m}{8\,m}\cdot 1\,m/s$$

$\boxed{\text{speed of boat} = \dfrac{\sqrt{65}}{8}\ m/s} \approx \boxed{\underline{1.01\ m/s}}$

@ $t=5$ sec:

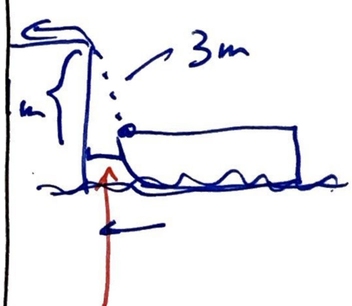
$\boxed{l_1 = l_0 - v\cdot t}$

$= \sqrt{65} - (1\,m/s)\cdot 5\,s = \sqrt{65}-5$

$\approx \underline{3m}$

$x_1 = \sqrt{l_1^2 - 1m^2}$

$= \sqrt{3^2-1^2} = \sqrt{8} \approx \underline{2.83m}$



$$\left.\frac{dx}{dt}\right|_{t=1} = \frac{l_1}{x_1}\left.\frac{dl}{dt}\right|_{t=0} = \left(\frac{3m}{2.83m}\right)1\frac{m}{s}$$

$\approx \boxed{1.1\ m/s}$ a little faster

⊛ Volume Rates

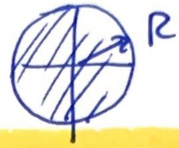
Volume of a sphere is

$V = \frac{4}{3}\pi R^3$

$A = 4\pi R^2$ surface area.

units: length m, area m^2, volume m^3

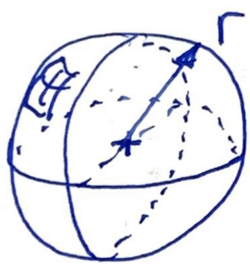
area of circle $A = \pi R^2$



EX The radius of a High Altitude Baloon, r, is increasing at 4mm/sec as it rises.

Q: When the balloon is 800mm, what is the rate of change of the volume of the balloon?

(i) Diagram



(ii) Data

$r = 800\,mm$ $\quad \frac{dr}{dt} = 4\,mm/s$

$\frac{dV}{dt} = ?$

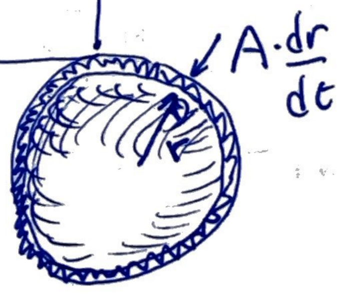
(iii) Eqn $\quad V = \frac{4}{3}\pi r^3$

(iv) Solve $\quad$ • Diff't $\Rightarrow \frac{d}{dt}\left(V = \frac{4}{3}\pi r^3\right)$

$\Rightarrow \frac{dV}{dt} = \frac{4\pi}{\cancel{3}} \cdot \cancel{3} r^2 \frac{dr}{dt}$

So $\boxed{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}}$

related rate eqn



• plug numbers in

$\frac{dV}{dt} = 4\pi(800m)^2 \cdot 4\frac{mm}{s}$

$= \boxed{16\pi\, 800^2\ mm^3/s}$

$= 1.024 \times 10^7 \pi\ mm^3/s$

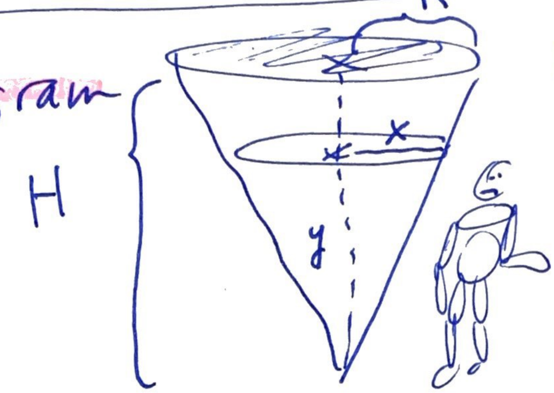
30 million mm^3/s

EX Inverted Conicle Reservoir

An inverted conical reservoir is being filled at the rate of 2 cu ft per min : $H = 10\text{ft}$, $R = 5\text{ft}$.

Q: How fast is the water level rising when it is 6 ft deep?

(i) Diagram



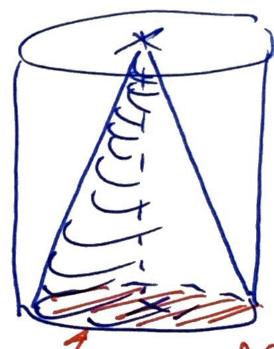
(ii) Data

$\frac{dy}{dt} = ?$, $y_0 = 6\text{ ft}$.

$\frac{dV}{dt} = 2\text{ ft}^3/\text{min}$

(iii) eqn

V of cone is $\frac{1}{3}$ the volume of the cyclinder the cone was delivered in



Area of Base

B

cylinder: $V = B^{\text{use}} \cdot H^{\text{eight}}$

$V_{cyl.} = (\pi R^2) H$

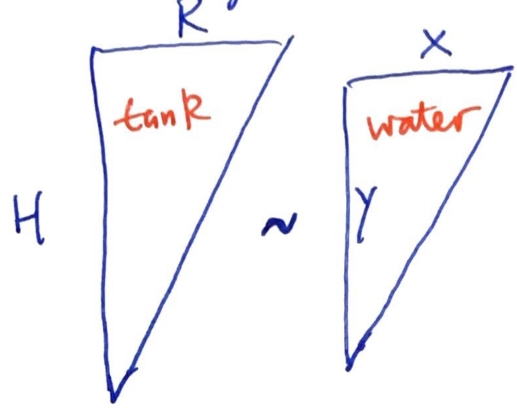
So

$$\boxed{V_{\substack{\text{cone}\\ \text{tank}}} = \frac{1}{3} \pi R^2 H}$$

then $\boxed{V_{\text{water}} = \frac{1}{3} \pi x^2 y}$

- We want to reduce the number of variables from 2 to 1:

- To do this we recognize that x & y are legs of a right triangle. This Δ is similar to the dimensions of the tank



- Similar Δ's : "H is to R as y is to x"

convert to eqn: $\dfrac{H}{R} = \dfrac{y}{x}$

- Since we want $\dfrac{dy}{dt}$ lets eliminate "x" from the volume eqn. So $\boxed{x = \dfrac{y \cdot R}{H}} = y\dfrac{5}{10} = \underline{\underline{y/2}}$

- Insert $x = y/2$ into the Volume formula

$$\Rightarrow V_{water} = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y$$

diff't...

$$\text{OR} \quad \boxed{V_{wate} = \frac{\pi}{12}y^3} \quad \& \quad \frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\left(\frac{dy}{dt}\right)$$

- Diff't $\boxed{\dfrac{dV}{dt} = \dfrac{\pi}{4} y^2 \dfrac{dy}{dt}}$ ← want

- populate $\dfrac{2\,ft^3}{min} = \dfrac{\pi}{4}(6\,ft)^2 \cdot \dfrac{dy}{dt}$

Solve for $\dfrac{dy}{dt}$

$$\Rightarrow \quad \frac{2 \cdot \cancel{4}^{2}}{\pi \cdot \cancel{36}\,\cancel{18}\,9} = \frac{dy}{dt}$$

ans. simplifies to... $\boxed{\dfrac{dy}{dt} = \dfrac{2}{9\pi}\ ft/min} \approx 0.07\ ft/m.$

about $\frac{1}{4}$ inch rise in 1 minute...

Math 211 5b Name ______

1. what is the rate of increase of the area of a square if it's edge is increasing at 2mm/s when the edge is 60mm?