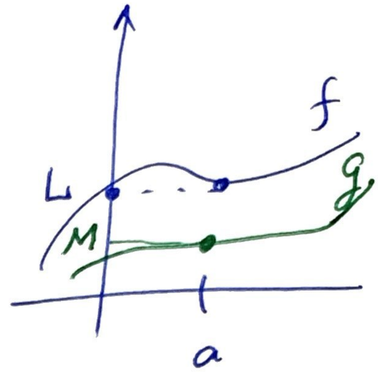
1.6 Limit Properties

I Limit Laws



- let $\lim\limits_{x\to a} f(x) = L$
- let $\lim\limits_{x\to a} g(x) = M$

sum/diff' rule: $\lim\limits_{x\to a} [f(x) \pm g(x)] = L \pm M$

scalar multiple rule: $\lim\limits_{x\to a} c f(x) = c \lim\limits_{x\to a} f(x) = cL$

limit product rule: $\lim\limits_{x\to a} [f(x) \cdot g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = L \cdot M$

limit quotient rule: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} f(x) \Big/ \lim\limits_{x\to a} g(x) = \dfrac{L}{M}, \ M \neq 0$

limit power rule: $\lim\limits_{x\to a} [f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n = L^n$

limit root rule: $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$

Ex evaluate this problem

$\lim_{x \to -2} \sqrt{x^4+3x+6}$, justify each step

$= \sqrt{\lim_{x \to -2} (x^4+3x+6)}$ root rule

$= \sqrt{\lim_{x \to -2} x^4 + \lim_{x \to -2} (3x) + \lim_{x \to -2} (6)}$ sum rule

$= \sqrt{\lim_{x \to -2} (x^4) + 3\lim_{x \to -2} (x) + \lim_{x \to -2} (6)}$ scalar mult. rule

$= \sqrt{\left(\lim_{x \to -2} x\right)^4 + 3\lim_{x \to -2} (x) + \lim_{x \to -2} (6)}$ exponent rule

$= \sqrt{(-2)^4 + 3(-2) + 6}$ evaluate the limit

$= \sqrt{16 \cancel{-6} \cancel{+6}}$ algebra

$= \sqrt{16}$

$= \boxed{4}$

* **IF** f(x) is a polynomial, or a product of poly's or a rational function $\left\{\frac{\text{polynomial}}{\text{polynomial}}\right\}$

then we can $\boxed{\text{just substitute "}a\text{" for } x}$

{So long as $\lim_{x \to a}$ (poly) exisists and in the case of a rational the denominator does not vanish. (go to zero)}

I.E. $\boxed{\lim_{x \to a} f(x) = f(a)}$

Ex Evaluate $\lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3)$

$$= \lim_{x \to -1} (x^4 - 3x) \cdot \lim_{x \to -1} (x^2 + 5x + 3)$$

$$= \left[(-1)^4 - 3(-1)\right] \cdot \left[(-1)^2 + 5(-1) + 3\right]$$

$$= [1 + 3] \cdot [1 - 5 + 3]$$

$$= (4) \cdot (-1)$$

$$= \boxed{-4}$$

Ex Evaluate $\lim_{x \to 4} \left(\dfrac{x^2 - 4x}{x^2 - 3x - 4} \right)$

- Can we just plug in "4"?

Try it $= \dfrac{(4)^2 - 4(4)}{(4)^2 - 3(4) - 4}$

$= \dfrac{16 - 16}{16 - 12 - 4}$

$= \dfrac{0}{0}$ oh oh ....

this invalidate the "theorem" just discussed because we are dividing by zero!

- Try to factor

$f(x) = \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

$\boxed{f(x) = \dfrac{x(x-4)}{(x-4)(x+1)}}$

cancel $(x-4)$

$\boxed{g(x) = \dfrac{x}{x+1}}$ which is identical to $f(x)$ everywhere but at $x = 4$

$g(4) = \dfrac{4}{4+1} = \boxed{\dfrac{4}{5}}$

Note:

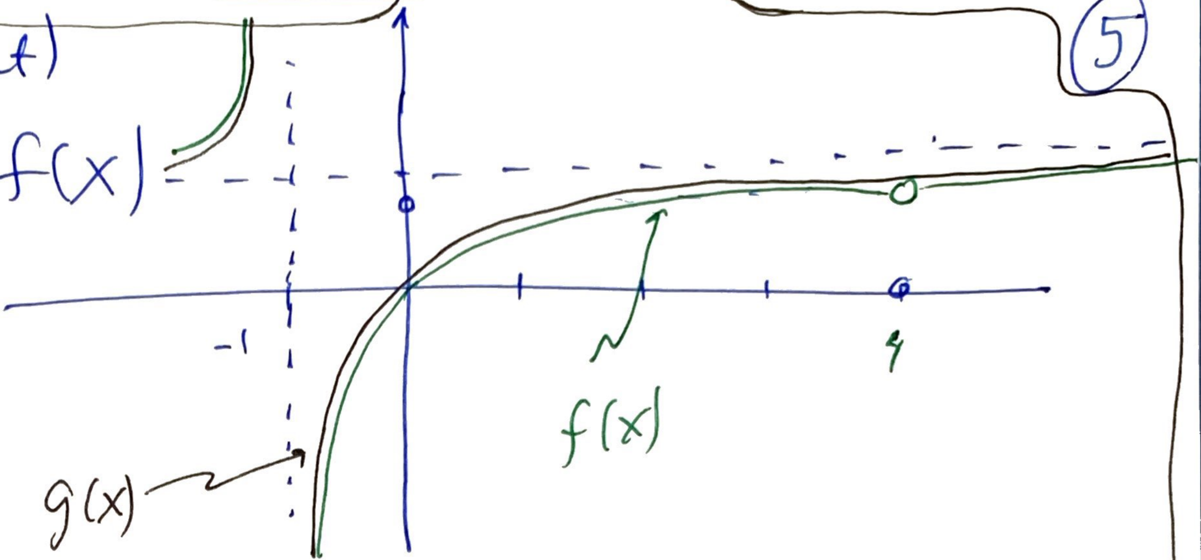
- $\lim_{x \to 4^-} f(x) = \dfrac{4}{5}$
- $\lim_{x \to 4^+} f(x) = \dfrac{4}{5}$

therefor $\lim_{x \to 4} f(x) = \dfrac{4}{5}$

So we can fill the hole

$\boxed{\text{Ex}}$ (cont)

- Graph $f(x)$



- precalc we just cancelled $(x-4)$ and kept going
- calc. we examine the limit from left & right and then if all is the same just keep going.

$D_f : \{x \mid x \neq 4\}$ ($x \neq -1$) $D_g : \{x \mid x \in \mathbb{R}\}$ (all real numbers except $x \neq -1$)

$R_f : \{y \mid y \neq 1\}$ "$y \neq 4/5$" $R_g : \{y \mid y \neq -1\}$

1.6 is finished

Math 211 $\boxed{\text{CW1b}}$ Name ______

1. Use $\left.\begin{aligned} x &= 0.1 \\ x &= 0.01 \\ x &= 0.001 \\ \text{and} \quad x &= -0.001 \\ x &= -0.01 \\ x &= -0.1 \end{aligned}\right\}$ to evaluate $f(x) = \left(\dfrac{3-|x|}{3+x}\right)$

$\boxed{\text{ANSWER}}$

$$\boxed{\lim_{x\to 0} \left(\frac{3-|0|}{3+0}\right) = 1}$$

EX Evaluate $\lim_{x \to -3} \left(\frac{3-|x|}{3+x} \right) \leftarrow f$

4

face value: $\frac{3-|-3|}{3+(-3)} = \frac{3-3}{3-3} = \frac{0}{0}$

So -3 is NOT in the domain of $\frac{3-|x|}{3+x}$.

Lets look at the one-sided limits

$$x \to -3^- \quad \text{and} \quad x \to -3^+$$

- $\lim_{x \to -3^-} \left(\frac{3-|x|}{3+x} \right)$

$= \lim_{x \to -3^-} \left(\frac{3-(-x)}{3+x} \right)$

$= \lim_{x \to -3^-} \left(\frac{\cancel{3+x}}{\cancel{3+x}} \right) = \boxed{1}$

Review

$$|x| = \begin{cases} +x & x \geq 0 \\ -x & x < 0 \end{cases}$$

when x is negative then negate the value

$|-3| = -(-3) = 3$

$|+3| = +(+3) = 3$

- $\lim_{x \to -3^+} \left(\frac{3-|x|}{3+x} \right)$

$= \lim_{x \to -3^+} \left(\frac{3-(-x)}{3+x} \right)$

$= \lim_{x \to -3^+} \left(\frac{\cancel{3+x}}{\cancel{3+x}} \right) = \boxed{1}$

since $\lim_{x \to -3^-} f = 1$

and $\lim_{x \to -3^+} f = 1$

Then

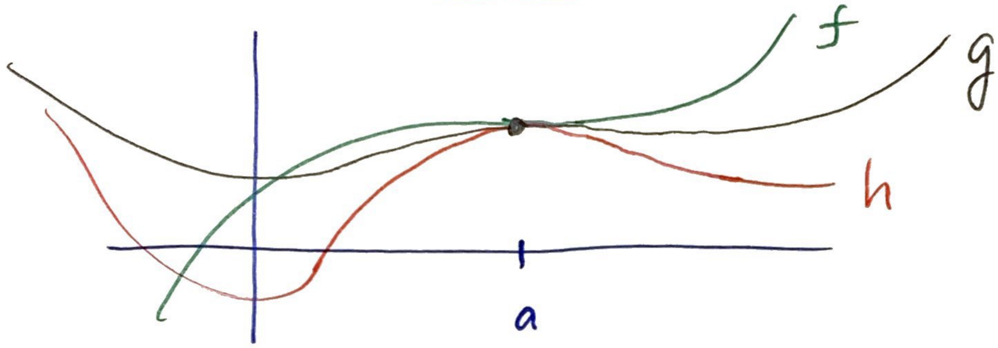
$$\boxed{\lim_{x \to -3} \left(\frac{3-|x|}{3+x} \right) = 1}$$

II Squeeze Thm

If $f(x) \le g(x) \le h(x)$ near $x = a$,
{except possibly at "a" itself}

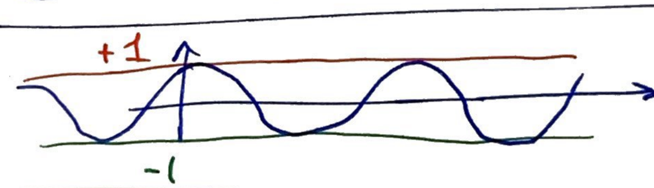
AND if $\boxed{\lim_{x \to a} f = L}$ and $\boxed{\lim_{x \to a} h = L}$ also

then $\boxed{\lim_{x \to a} g(x) = L}$ too



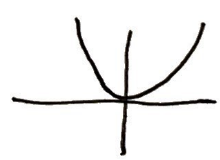
EX Eval $\lim_{x \to 0} \left(x^2 \cos(x) \right)$ using the range of cos

$R_{y=\cos(x)} : \{ y \mid |y| \le 1 \}$



BTW $|y| \le 1$ is a shortcut to $-1 \le y \le 1$

- Near $x=0$ x^2 is positive



$x^2 * \left[\boxed{-1 \le \cos(x) \le +1} \right]$

$\Rightarrow \quad -x^2 \le x^2 \cos(x) \le x^2$

$\downarrow \qquad\qquad \downarrow$

0 by squeeze thm $x^2\cos(x) \to 0$ 0

$\boxed{\lim_{x \to 0} x^2 \cos(x) = 0}$