1.5 Limits of a function

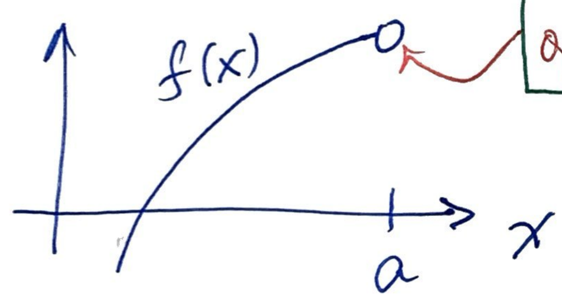
The function $f(x)$ is defined @ $x = a$ if we can calculate $f(a)$.

Ex $\tan\left(\frac{\pi}{4}\right) = 1$ $\tan\left(\frac{\pi}{2}\right) = \text{undefined}$ ∞

tan is defined @ $\frac{\pi}{4}$ but not $\frac{\pi}{2}$.

- It is possible that we cannot evaluate a function at $x = a$ but we will see that as we get closer to $x = a$ we can see the function's value converge.

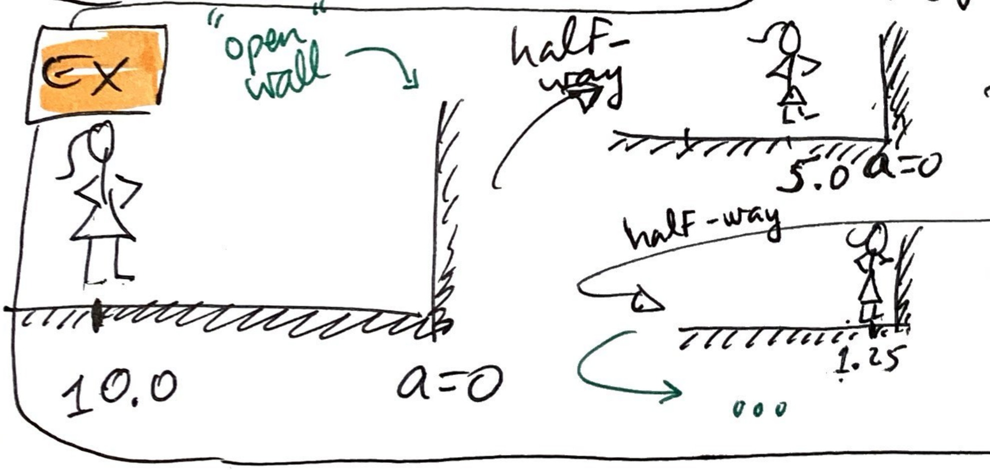
Ex



open interval at $x = a$

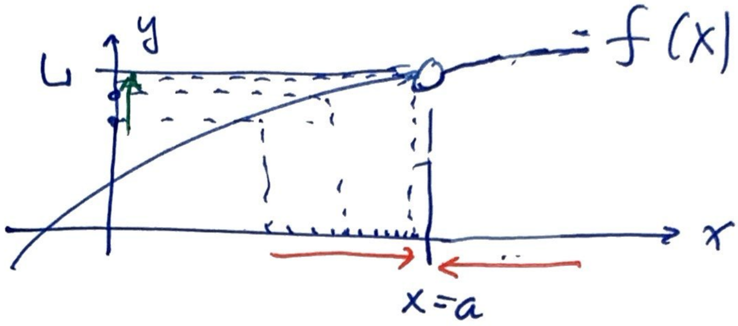
- we can get closer and closer but never touch $x = a$

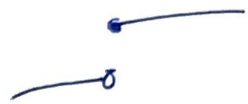
Ex



you never touch the wall!

Def: The limit of a function $f(x)$ at $x=a$ is L if, as x approaches a, $f(x)$ approaches L





$$\boxed{\lim_{x \to a} f(x) = L}$$

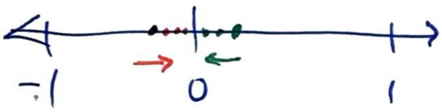
Unless otherwise indicated this limit must exist from either side of $x=a$

Ex Estimate the limit as $x \to 3$ for $f(x) = \dfrac{e^{-4x} - 1}{x}$.

$$f(3) = \frac{e^{-4(3)} - 1}{3} = -0.333331285$$

Ex Estimate the limit as $x \to 0$ (move in from both sides of zero)

x	$f(x) = \dfrac{e^{-4x} - 1}{x}$	
0.1	$f(0.1) = [\exp(-4(0.1)) - 1]/0.1$	$=$ $\boxed{-4.918}$
0.01	$f(0.01) = [\exp(-4(0.01)) - 1]/0.01$	$=$ $\boxed{-4.081}$
0.001	$f(0.001) = [\exp(-4(0.001)) - 1]/0.001$	$=$ $\boxed{-4.008}$
-0.001	$f(-0.001) = [\exp(-4(-0.001)) - 1]/(-0.001)$	$=$ $\boxed{-3.992}$
-0.01	$f(-0.01) = [\exp(-4(-0.01)) - 1]/(-0.01)$	$=$ $\boxed{-3.921}$
-0.1	$f(-0.1) = [\exp(-4(-0.1)) - 1]/(-0.1)$	$=$ $\boxed{-3.297}$



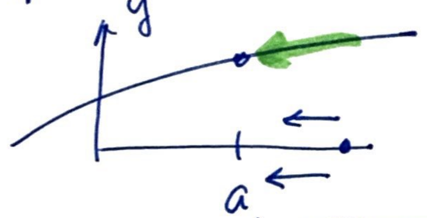
The estimate of $\lim_{x \to 0} f(x)$ is $\boxed{-4.00}$

Soon we will introduce the derivative of a function $f(x)$, evaluated at $x = a$, as

$$\frac{\Delta f}{\Delta x} \to \boxed{\frac{df}{dx} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \begin{array}{l} \to 0 \\ \hline \to 0 \end{array}$$

For $f(x)$ to have a limit as $x \to a$ then we say $f(x)$, as $x \to a^-$, must equal $f(x)$ as $x \to a^+$

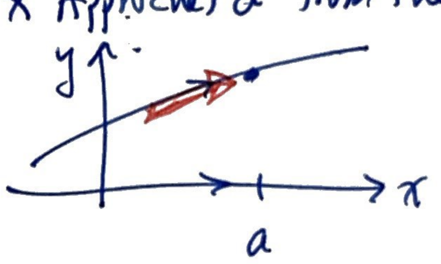
x Approaches "a" from the right



(+) side

$$\boxed{\lim_{x \to a^+}}$$

x Approches "a" from the Left



$$\boxed{\lim_{x \to a^-}}$$

Then a two sided limit exists if, by definition

we have $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

or we just say " the $\boxed{\text{limit exist}}$ "

- Notationally

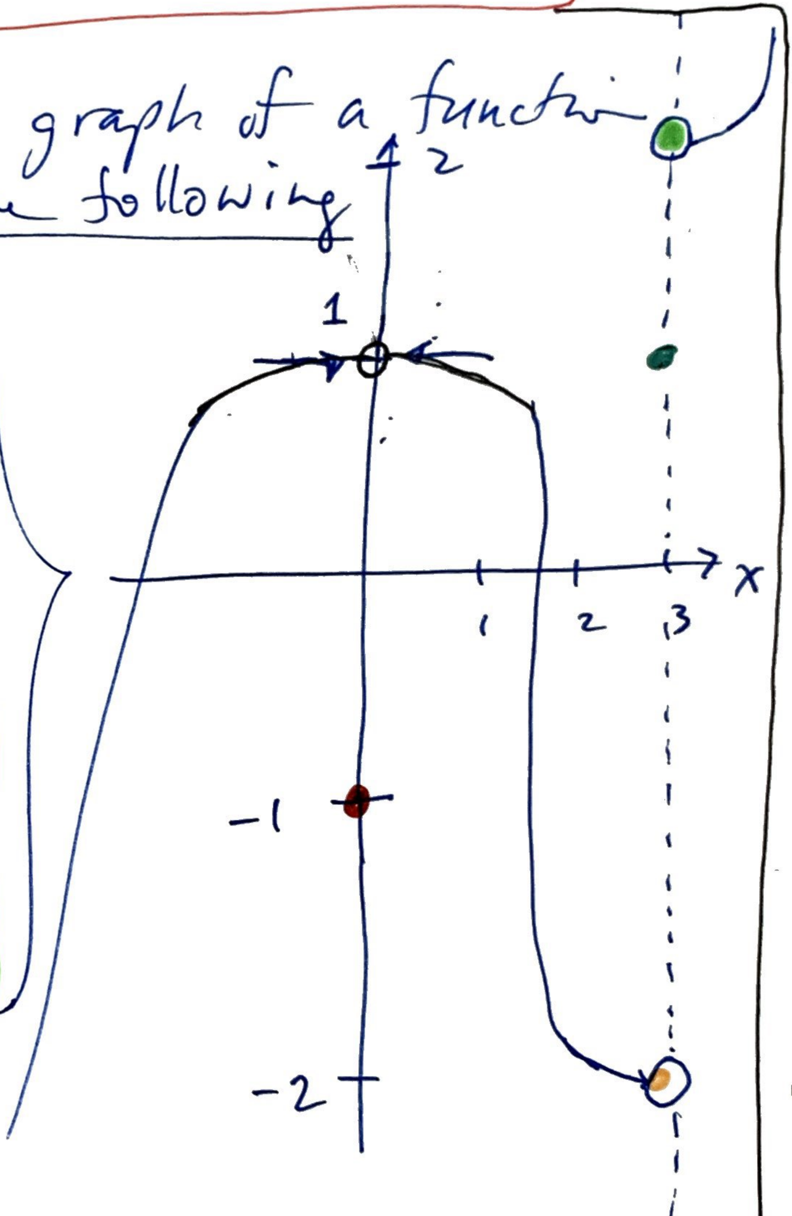
$$\boxed{\lim_{x \to a} f(x) = L \text{ means both } \lim_{x \to a^-} f(x) = L \ \& \ \lim_{x \to a^+} f(x) = L}$$

II One-sided Limits

- Left-handed limit: $\boxed{\lim_{x \to a^-} f(x) = L}$
- RH limit: $\boxed{\lim_{x \to a^+} f(x) = M}$

EX Sketch a possible graph of a function that satisfies the following

- $\boxed{f(0) = -1}$
- $\boxed{f(3) = 1}$
- $\boxed{\lim_{x \to 0} f(x) = 1}$
- $\boxed{\lim_{x \to 3^-} f(x) = -2}$
- $\boxed{\lim_{x \to 3^+} f(x) = +2}$

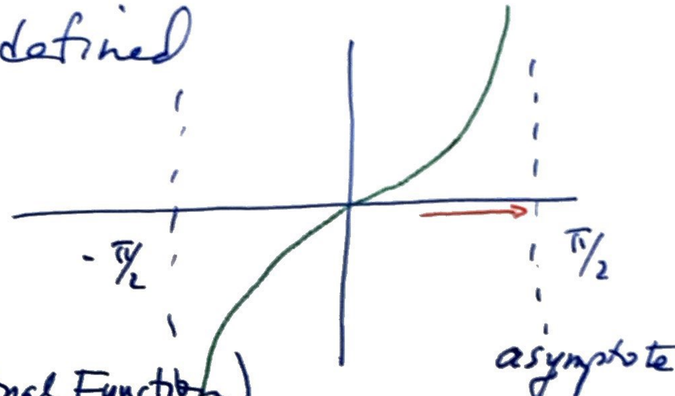


III Infinit Limits

Here $\boxed{\lim_{x \to a^-} f(x) = \text{undefined}}$

$= \pm\infty$ ← math ppl don't like this notation

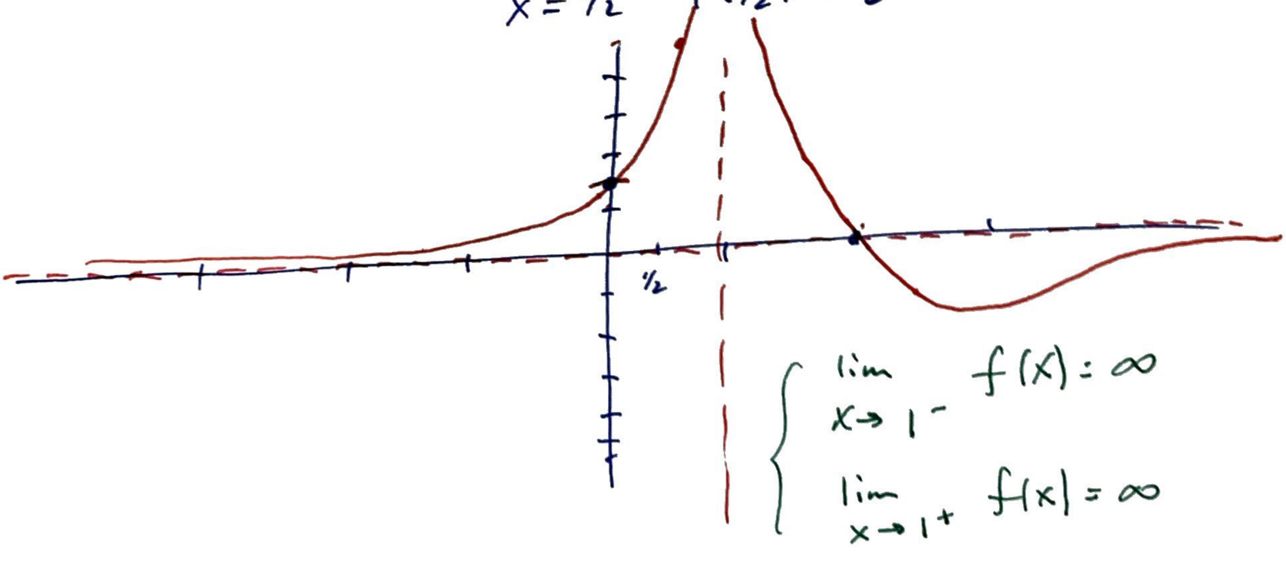
Ex $\lim_{x \to \pi/2} \tan(x) = \text{undefined}$



Ex Vertical Asymptoter (Rational Function)

sketch $f(x) = \dfrac{2-x}{(x-1)^2}$ ← deg = 1 ← deg = 2

- Horiz. Asymptote @ $x = 0$ since degree top < deg. bot
- vertical Asymp. @ $x = 1$ { even asymp. since $(x-1)^2$ }
- Helper point : $x = 0$ $f(0) = 2$
  $x = 2$ $f(2) = 0$
  $x = 1/2$ $f(1/2) = 6$



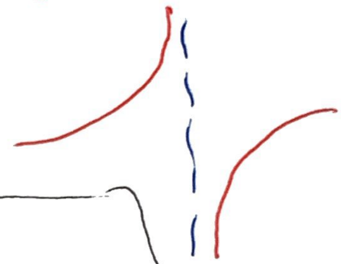
$$\begin{cases} \lim_{x \to 1^-} f(x) = \infty \\ \lim_{x \to 1^+} f(x) = \infty \end{cases}$$

⊛ ODD Asymptotes

$$\lim_{x\to a^-} f(x) = +\infty$$

$$\lim_{x\to a^+} f(x) = -\infty$$

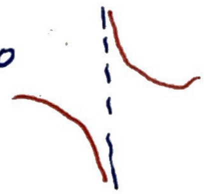
odd asymptote



n = odd

or vice-versa

$$\boxed{\lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = +\infty}$$



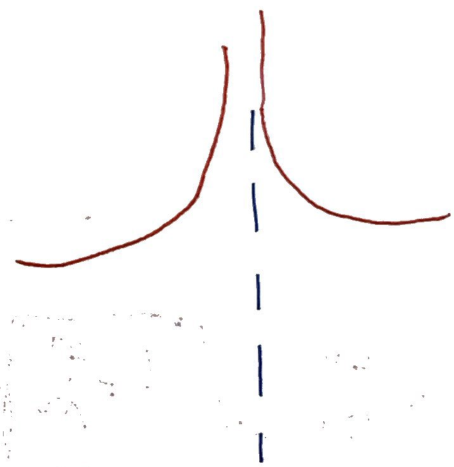
n = odd

denom $\dfrac{}{(x-a)^n}$

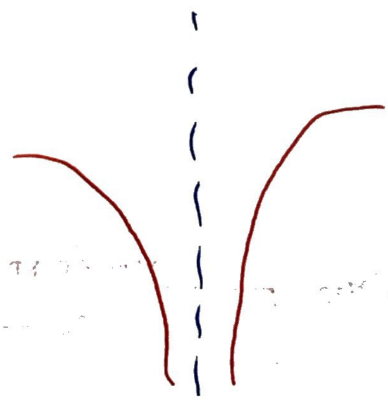
⊛ EVEN Asymptotes

$$\lim_{x\to a^-} f(x) = \pm\infty$$

$$\lim_{x\to a^+} f(x) = \pm\infty$$



or

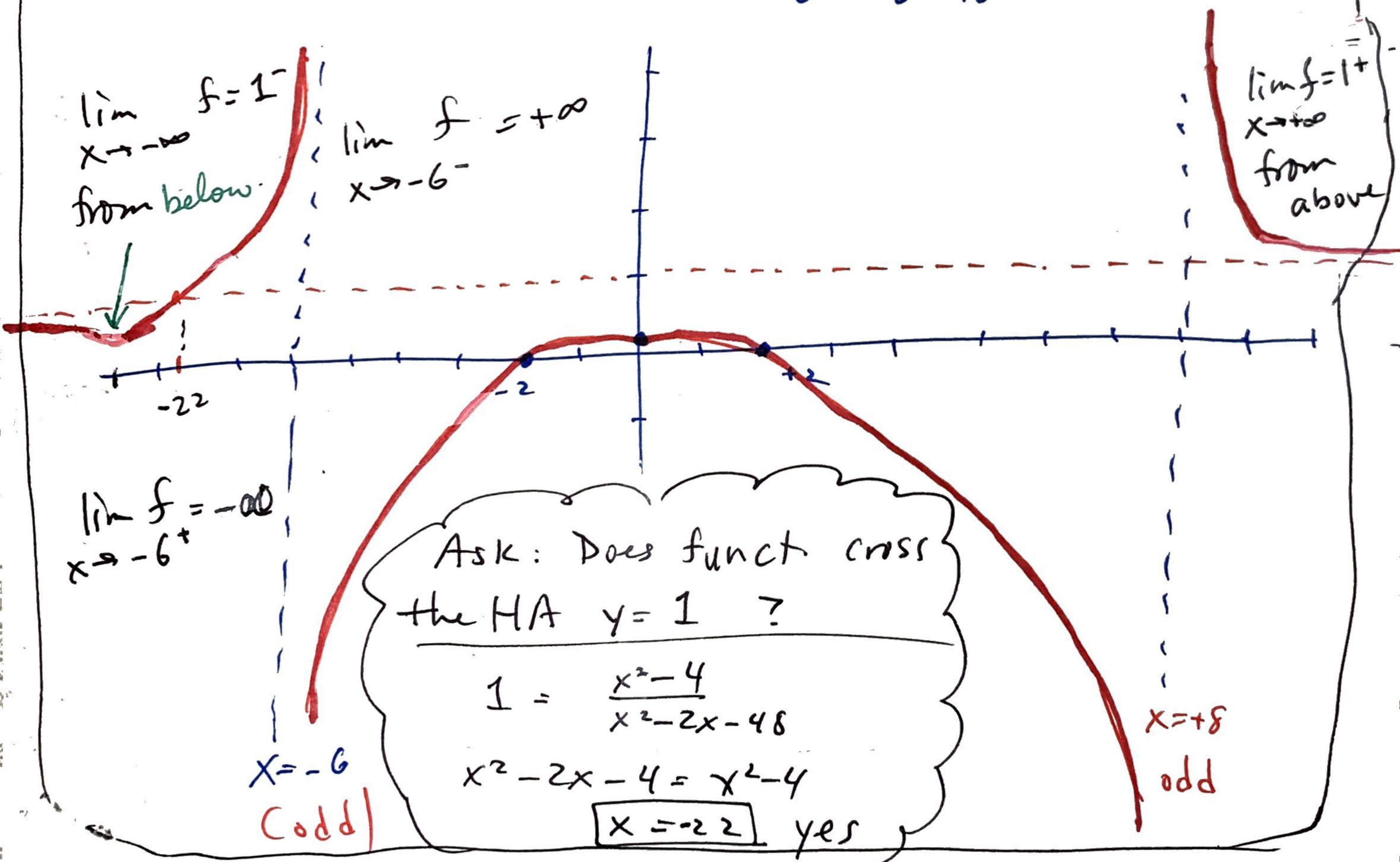


Ex Analyze $f: y = \dfrac{x^2-4}{x^2-2x-48}$

(i) factor (if possible) $y = \dfrac{(x-2)(x+2)}{(x-8)(x+6)} = \dfrac{x^2-4}{x^2-2x-48} \approx \dfrac{x^2}{x^2} = 1$

(ii) Steps before graphing:

- HA: $\dfrac{x^2}{x^2} = 1$ deg top = deg bot
- VA: $x = 8$ odd, $x = -6$ odd
- ~~zeros~~: $x = \pm 2$ ~~numerator~~ $= 0$
- help: $x = 0$ $y = \dfrac{0^2-4}{0^2-2\cdot0-48} = +\dfrac{1}{12}$



Ask: Does funct cross the HA $y = 1$?

$1 = \dfrac{x^2-4}{x^2-2x-48}$

$x^2-2x-4 = x^2-4$

$\boxed{x = -22}$ yes