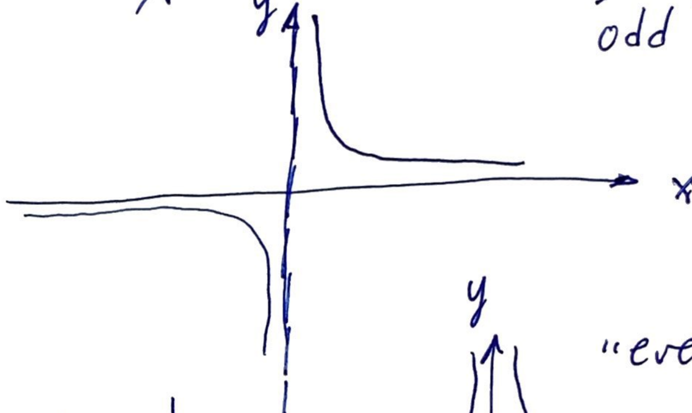
$\boxed{3.7}$ Graphing Rational Functions

- A rational number is a number that can be written as fraction : $\frac{a}{b}$
- A rational function is function composed of two polynomial functions in quotient form

$$R(x) = \frac{f(x)}{g(x)}$$ polynomials.

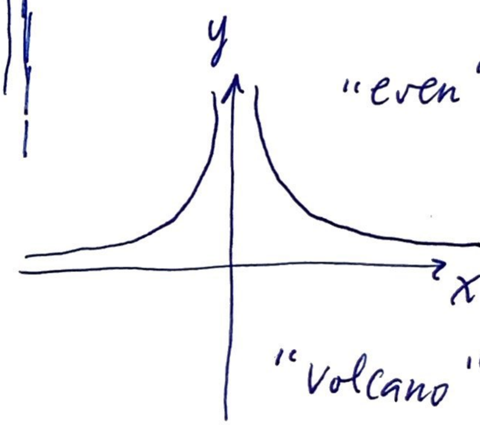
* Consider graphing $R(x) = \frac{1}{x}$ "reciprocal function"

"odd"



next consider $R(x) = \frac{1}{x^2}$

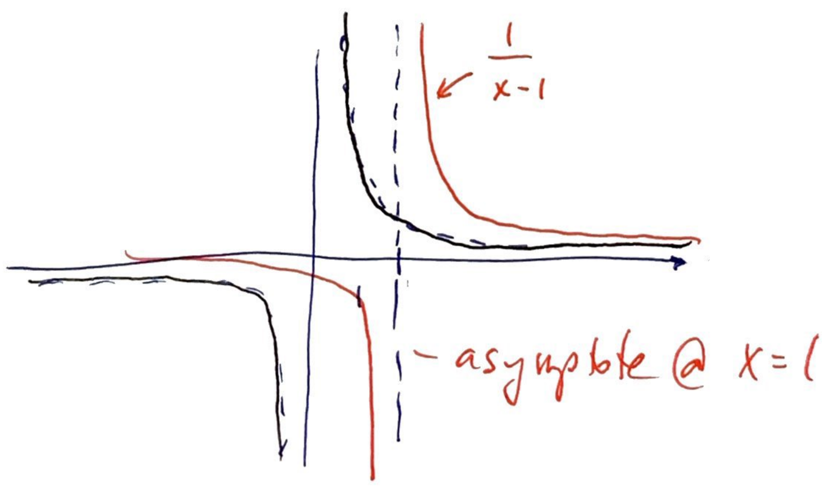
"even"



"volcano"

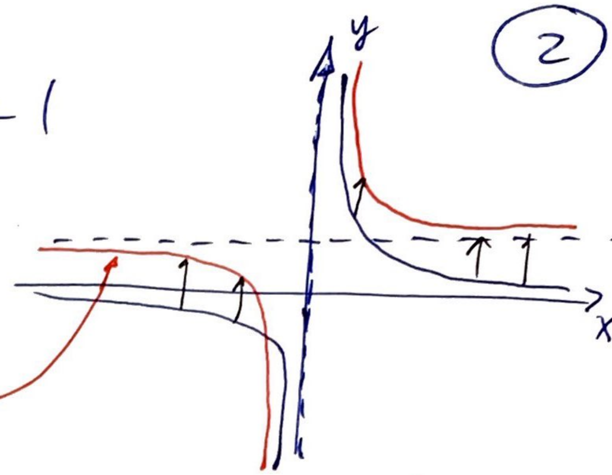
* Transformations

$$H(x) = \frac{1}{x-1}$$



* Vertical Asymptotes

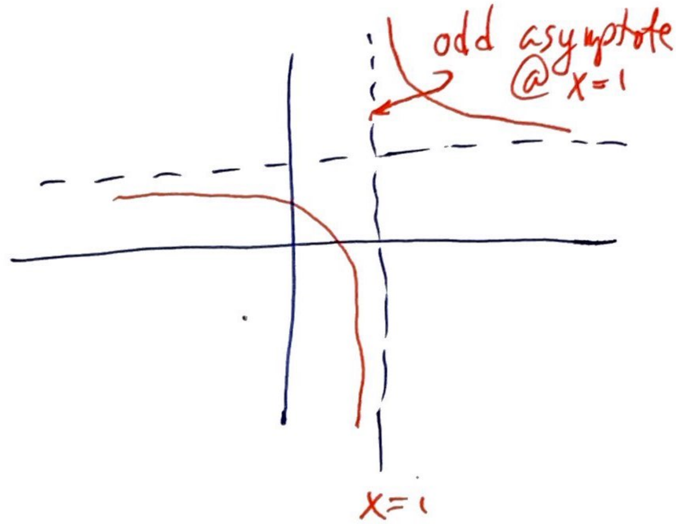
- vertical shifts $G(x) = \frac{1}{x} + 1$



common denominator

$$G(x) = \frac{1}{x} + \frac{x}{x} = \boxed{\frac{x+1}{x}}$$

- Together $F(x) = \frac{1}{x-1} + 1$



common denom.

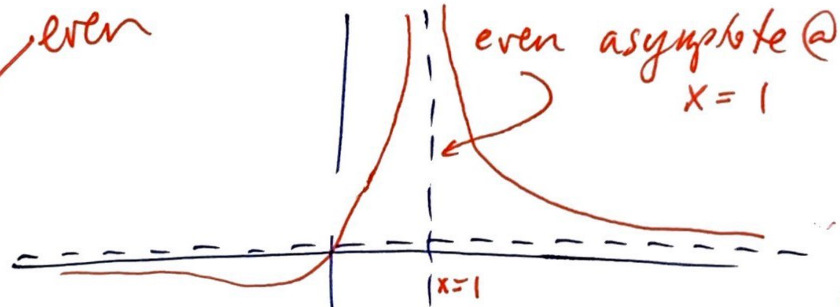
$$F(x) = \frac{1}{x-1} + \frac{x-1}{x-1}$$

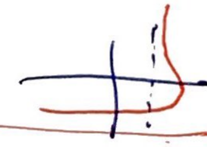
$$= \frac{1+x-1}{x-1}$$

$$\boxed{F(x) = \frac{x}{(x-1)^1}}$$ odd

Vert. Asympt is when Denom = 0

- $J(x) = \frac{x}{(x-1)^2}$ even



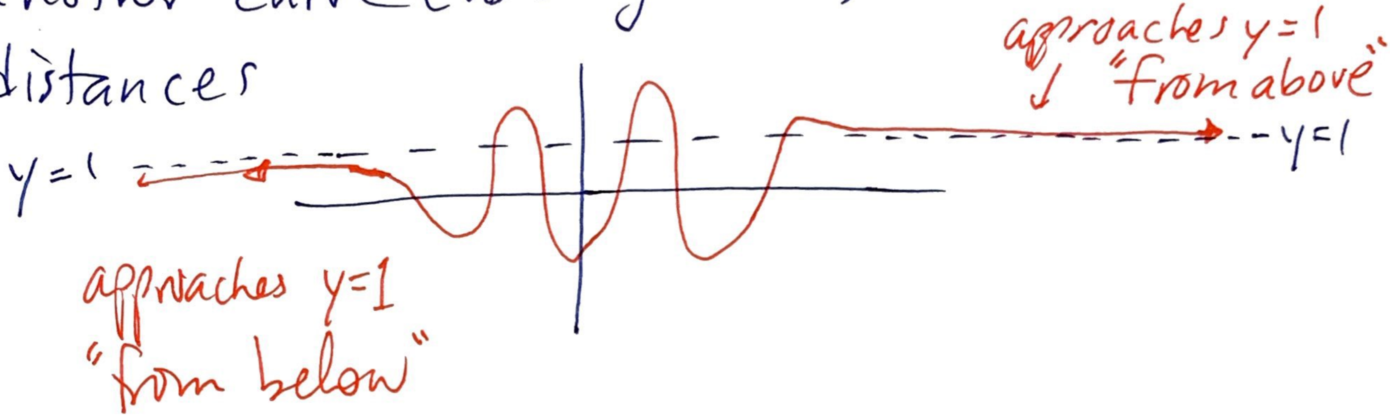


Curves cannot pass through vertical asymptotes {fails the vertical line test}

* As you will see a curve can cross a horizontal asymptote

* Horizontal Asymptotes

H.A. occur when a curve approaches another curve (usually a line) at extreme distances



Q: How do we know what the distant behavior is?

A: long divide or divide numerator & denominator by powers of x.

$\boxed{\text{EX}}$ $F(x) = \dfrac{x+2}{x-1}$ note: limit as $x \to \infty$ of $\frac{1}{x}$ is 0

now divide top & bottom by x: $\dfrac{(x+2)\cdot\frac{1}{x}}{(x+1)\cdot\frac{1}{x}}$

$$F(x) = \frac{\frac{x+2}{x}}{\frac{x-1}{x}}$$

$$F(x) = \frac{1+\frac{2}{x}}{1-\frac{1}{x}}$$

note that the $\displaystyle\lim_{x\to\infty}\left(\frac{1+\cancelto{0}{\frac{2}{x}}}{1-\cancel{\frac{1}{x}}}\right) = \frac{1}{1} = \underline{\underline{1}}$ H.A.

* long divide

4

$$F(x) = \frac{x+2}{x-1}$$

$$x-1 \overline{\smash{)}\, x+2} \quad 1 \text{ r } 3$$
$$-(x-1)$$
$$3$$

$$F(x) = 1 + \frac{3}{x-1} \to 0$$

$$\boxed{\lim_{x\to\infty} F(x) = 1}$$

$$\boxed{\lim_{x\to-\infty} F(x) = 1}$$

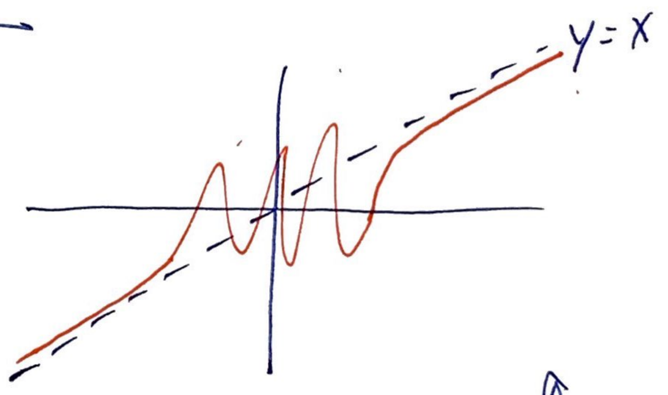
$y = 1$ H.A.

* slant asymptotes

$\boxed{\text{EX}}$ Find the H.A. of $R = \dfrac{x^3+2x-1}{x^2+x-1}$

- divide by x^2 (the highest term on the denom)

$$R = \frac{\frac{x^3}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2} - \frac{1}{x^2}}$$



$$R = \frac{x + \frac{2}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}}$$

for large x, $\frac{2}{x}$ & $\frac{1}{x^2}$ & $\frac{1}{x}$ & $\frac{1}{x^2}$ "vanish"

for large x $R \approx \dfrac{x+0-0}{1+0-0} = \boxed{x}$ "slant asymptote

* oblique asymptote

$$H(x) = \frac{x^4 + 4x^3 - 2x + 3}{x^2 + 1}$$

Long divide

$$
\begin{array}{r}
x^2 + 4x - 1 \quad r\ -6x+4 \\
x^2+0x+1 \,\big)\, \overline{x^4 + 4x^3 + 0x^2 - 2x + 3} \\
-(x^4 + 0x^3 + 1x^2) \\
\hline
4x^3 - x^2 - 2x \\
-(4x^3 + 0x^2 + 4x) \\
\hline
-x^2 - 6x + 3 \\
-(-x^2 + 0x - 1) \\
\hline
-6x + 4
\end{array}
$$

$$\boxed{H(x) = x^2 + 4x - 1 + \frac{-6x+4}{x^2+1}}$$

As x gets large the term $\boxed{\frac{-6x+4}{x^2+1}} \to 0$

$$\frac{-6x+4}{x^2+1} = \frac{-\frac{6x}{x^2} + \frac{4}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = \frac{-\frac{6}{x} + \frac{4}{x^2}}{1 + \frac{1}{x^2}} \to 0$$

$\div x^2$

$-\frac{6}{x} \to 0$, $\frac{4}{x^2} \to 0$, $\frac{1}{x^2} \to 0$

When the degree of x in the numerator is less than the degree of x in the denominator then the expression vanishes @ large x.

$H(x) \to x^2 + 4x - 1$ as $x \to \pm\infty$ "oblique asymptote"

* zeros of rational functions.

$R(x) = \dfrac{f(x)}{g(x)}$ then the zeros are when $f(x)=0$ unless $g(x)=0$ at the same x.

$\boxed{\text{EX}}$ $R(x) = \dfrac{1}{x-1} + 3 \cdot \dfrac{x-1}{x-1} = \dfrac{3x-2}{x-1}$

$= \dfrac{1+3(x-1)}{x-1}$

The numerator is zero when $3x-2=0$ or $\boxed{x = 2/3}$

* Procedures for sketching rational functions:

1. Find y intercept (let $x=0$)
2. factor numerator & denominator {eliminate common factors above and below}
3. zeros occur when the numerator $=0$
4. Vertical Asymptotes occur when the denominator $=0$
5. take note of the evenness or oddness of the VA
6. determine the HA, slant or oblique nature of the far away behavior.
7. Determine if the curve crosses the HA, slant or oblique asymptote
8. Sketch the graph (may need some C.S.I.)

9. Helper points.

EX Sketch $f(x) = \dfrac{x}{x^2-9}$

1. y-int. $f(0) = \dfrac{0}{0^2-9} = 0$ $(0,0)$

2. factor $f(x) = \dfrac{x}{(x+3)^1(x-3)^1}$

3. zeros @ $\underline{\underline{x=0}}$

4. VA: $x=-3$ (odd) & $x=+3$ (odd)

5.

6. HA: • if degree of num. > degree of denom.
$\boxed{\text{long divide}}$ to get HA, slant or oblique
• if degree of num = deg. of the denom
then HA is a line $\boxed{y = \dfrac{a_n}{b_n}}$
where $\dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots}{b_n x^n + b_{n-1}x^{n-1} + \cdots}$
• if degree of num < deg. of denom
HA is $\boxed{y=0}$

here $\underline{\underline{y=0}}$ is the HA

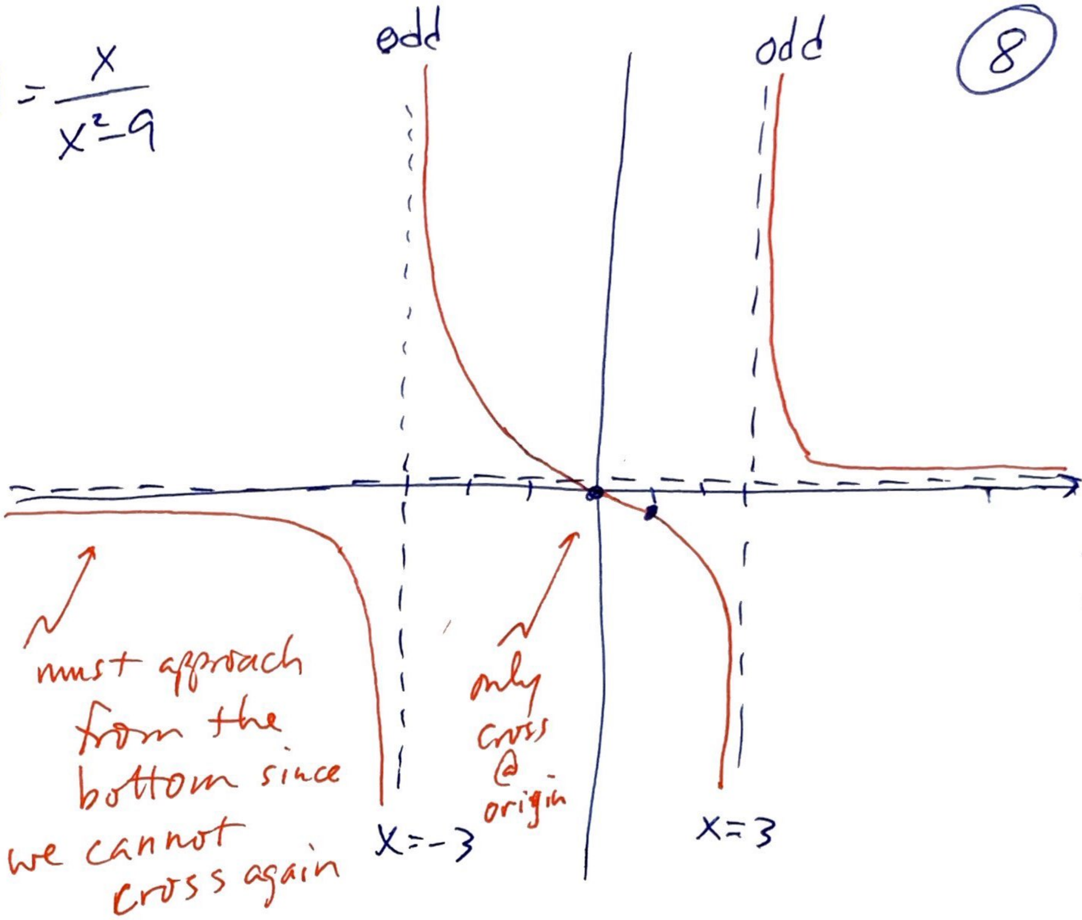
7. Does $f(x)$ cross $y=0$? HA $\dfrac{x}{x^2-9} = 0$, f(x) y=0 ans: yes $\underline{\underline{@\ x=0}}$

8. Sketch

Sketch of $f(x) = \frac{x}{x^2-9}$ (8)

Helper Point ...

$f(1) = \frac{1}{1^2-9} = \frac{-1}{8}$



$\boxed{\text{Ex}}$ $f(x) = \frac{x^2+x+6}{x^2-10x+24}$

3. x-int: $x^2+x+6=0 \rightarrow (x\quad)(x\quad)$? No

$x = \frac{-1 \pm \sqrt{1^2-4\cdot1\cdot6}}{2\cdot1}$ complex: No zeros.

1. y-int $f(0) = \frac{0^2+0+6}{0^2-10\cdot0+24} = \frac{1}{4}$ $(0, \frac{1}{4})$

2. factor denom?

$x^2-10x+24 = (x-\cancel{2}^{6})(x-\cancel{12}^{4})$

4 & 5. VA $x=6$ (odd) & $x=4$ (odd)

6. HA deg on top = deg on botton so $y = \frac{1\cdot x^2}{1\cdot x^2} = 1$

f(x) → ↙ HA

7. $\frac{x^2+x+6}{x^2-10+24} \stackrel{?}{=} 1$

$^2+x+6 = -10x+24 \rightarrow 11x = 18$

$\boxed{x = \frac{18}{11}}$

⑨

* sketch $f(x) = \dfrac{x^2+x+6}{(x-4)^1(x-6)^1}$

odd odd

• Helper point

$f(3) = \dfrac{9+3+6}{9-30+24}$

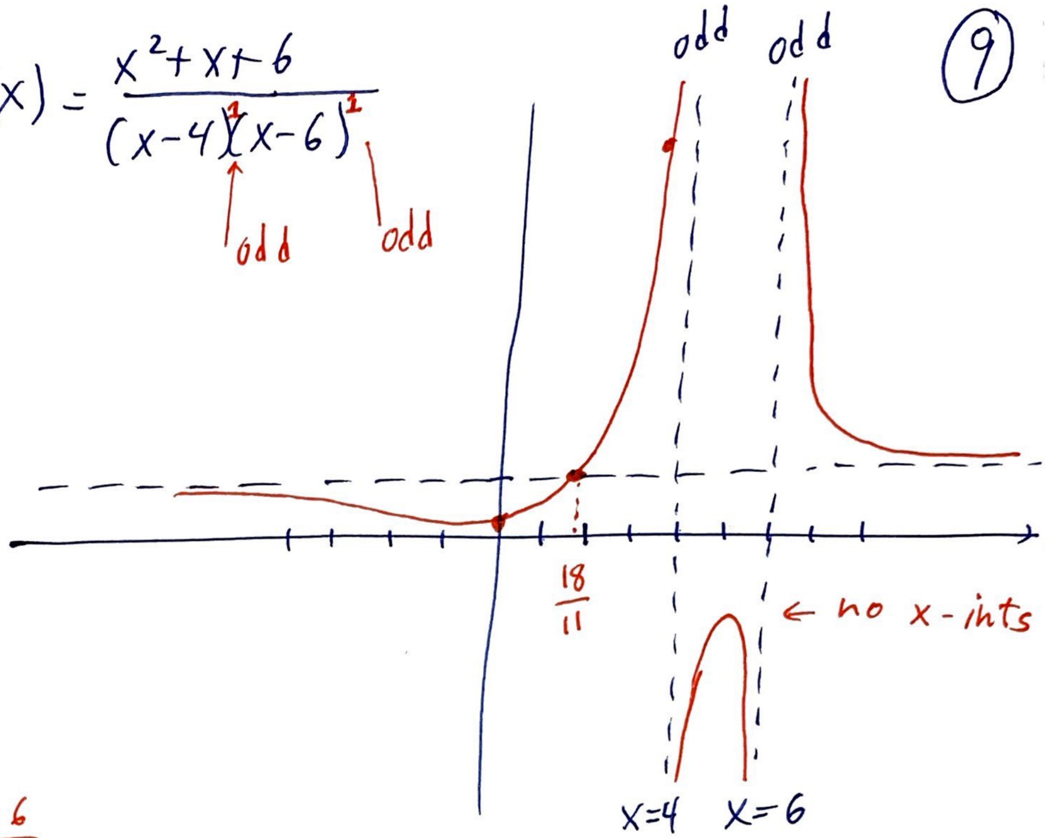
$= \dfrac{18}{3}$

$= 6$

$(3,6)$

• no zeros

• $f(5) = \dfrac{25+5+6}{(5-4)(5-6)} = \dfrac{36}{-1}$



EX Sketch $w(x) = \dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$ ⑩

(a) ~~zeros~~: $x = 1, -3, 5$ all line-like

(b) y-int : $w(0) = \dfrac{(-1)\cdot(3)\cdot(-5)}{(2)^2(-4)} = \boxed{\dfrac{15}{-16}} \approx -1$

(c) VA: $x = -2$ (even) & $x = 4$ (odd)

(d) HA: deg on top $= x^3$, deg on bott $= x^3$, $y = \dfrac{1x^3+\dots}{1x^3+\dots}$

$\boxed{y = \frac{1}{1} = 1}$

(e) Crossings: $\dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)} \overset{?}{=} 1$

$\Rightarrow (x^2-2x-3)(x-5) \overset{?}{=} (x^2+4x+4)(x-4)$

$\cancel{x^3} - 2x^2 - 3x - 5x^2 + 10x + 15 \overset{?}{=} \cancel{x^3} + 4x^2 + 4x - 4x^2 - 16x - 16$

$-7x^2 + 7x + 15 \overset{?}{=} -12x - 16$

$-7x^2 + 19x + 31 \overset{?}{=} 0$?

$7x^2 - 19x - 31 = 0$

$x = \dfrac{19 \pm \sqrt{19^2 + 4\cdot 7\cdot 31}}{2\cdot 7}$

$x \approx \dfrac{19 \pm 35}{14} \approx -1.1, 3.8$??

Mistake