9.8 Determinants and Cramer's Rule

Recall in 9.7 we found a formula for the inverse of a 2X2. In the denominator we saw a common number D, which we called the determinant.

2-D: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then it's determinant is $D = ad - cb$

Ex $\begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} = 5 \cdot 1 - (2)(-6) = \underline{\underline{17}}$

- The determinant of a matrix is a single scalar number.

Notation:

$\det(A)$ or $\begin{vmatrix} \# & \# \\ \# & \# \end{vmatrix}$

Not $\begin{bmatrix} \# & \# \\ \# & \# \end{bmatrix}$ as this is just a matrix

nor $\det \begin{vmatrix} \# & \# \\ \# & \# \end{vmatrix}$ as this is duplicating meaning.

- For 3-D, if we went through the same process we used to solve for the 2x2 inverse formula,

namely solve for $j, k, l, m, n, o, p, q, r$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we would see that the denominators of $j, k, l, \dots, p, q, r$ are all the same.

- That denom is the Determinat of a 3X3.

- Fortunately we have a short cut in finding the determinat of a 3X3 matrix

- $\boxed{\text{3X3 only:}}$

let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then

(ii) multiply up the antidiagonals

(iii) mult. up the daig'ls

(iv)

$$\boxed{\det(A) = [aei + bfg + cdh] - [gec + hfa + idb]}$$

$$\text{So } \det(A) = \left| \begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array} \right.$$

(i) Repeat columns one & two

Ex Find the determinant of

$$[-45 + 3 + 12] - [-9 - 30 - 6]$$

$$\left|\begin{array}{ccc|cc} 5 & 1 & -1 & 5 & 1 \\ 2 & 3 & 1 & 2 & 3 \\ 3 & -6 & -3 & 3 & -6 \end{array}\right|$$

$$= [-30] - [-45]$$

$$= \boxed{15}$$

Cramers Rule for 2X2 Systems:

The solution of the system

$$ax + by = e$$
$$cx + dy = f$$

is

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Ex Solve $\begin{aligned} x-3y &= 4 \\ 2x+y &= 1 \end{aligned}$ via Cramer's

$$x = \frac{\begin{vmatrix} 4 & -3 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix}} \quad , \quad y = \frac{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix}}$$

$$= \frac{7}{7} \qquad\qquad = \frac{-7}{7}$$

$$= 1 \qquad\qquad = -1$$

$$\boxed{(x,y) = (1,-1)}$$

⊛ Cramer's Rule for 3-D System:

The solution of
$$\begin{aligned} ax+by+cz &= j \\ dx+ey+fz &= k \\ gx+hy+iz &= l \end{aligned}$$

is
$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} & " & \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} & " & \end{vmatrix}}$$

So we need to evaluate <u>four 3x3</u> matrices. Alot of work. Use the Gauss-Jordan (most efficient) or the Inverse matrix method (less efficient)

Ex Solve via Cramer's (set-up only) 6

$$\begin{aligned} -5x + 2y - 4z &= -3 \\ 4x - 3y - z &= -1 \\ 3x - 3y + 2z &= 0 \end{aligned}$$

$$x = \frac{\begin{vmatrix} -3 & 2 & -4 \\ -1 & -3 & -1 \\ 0 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} -5 & 2 & -4 \\ 4 & -3 & -1 \\ 3 & -3 & 2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} -5 & -3 & -4 \\ 4 & -1 & -1 \\ 3 & 0 & 2 \end{vmatrix}}{\text{ditto}} \qquad z = \frac{\begin{vmatrix} -5 & 2 & -3 \\ 4 & -3 & -1 \\ 3 & -3 & 0 \end{vmatrix}}{\text{ditto}}$$