9.7 Inverse Matrices

Recall we saw

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} d \\ e \end{pmatrix}}_{\vec{b}}$$

← abbreviation for system.

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\begin{aligned} ax+by &= d \\ cx+dy &= e \end{aligned}}$$

Ex write in matrix form

$$\begin{aligned} 3x + y - 2z &= 2 \\ x - 2y + z &= 3 \\ 2x - y - 3z &= 3 \end{aligned}$$

$$\begin{pmatrix} 3 & 1 & -2 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

- coefficient matrix
- variable vector
- const. vector

$$\mathbb{A} = \begin{pmatrix} 3 & 1 & -2 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

- matrix eqn

$$\boxed{\mathbb{A}\vec{x} = \vec{b}}$$

* Inverse of a matrix (square matrices only)

Def If $\mathbb{A}\mathbb{B} = \mathbb{I}$ then $\mathbb{B}$ is the inverse matrix of $\mathbb{A}$ and $\mathbb{A}$ is the inverse matrix of $\mathbb{B}$. $\mathbb{I}$ is called the identity matrix since $\mathbb{I}\mathbb{A} = \mathbb{A}$, $\mathbb{A}\mathbb{I} = \mathbb{A}$

$$\mathbb{I}_{1\times1} = [1], \quad \mathbb{I}_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbb{I}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ etc.}$$

Ex Are $\mathbb{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $\mathbb{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ inverses?

$$\mathbb{A}\mathbb{B} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1 + 1\cdot0 & 1\cdot0 + 0\cdot1 \\ -1\cdot1 + 1\cdot1 & -1\cdot0 + 1\cdot1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yes.

- Inverse matricas only exist for $\boxed{\text{square}}$ matrices
- Notation: If $\mathbb{B}$ is the inverse of $\mathbb{A}$ we write $\boxed{\text{"}\mathbb{A}^{-1}\text{"}} = \mathbb{B}$. $\neq \frac{1}{\mathbb{A}}$
- $\boxed{\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}}$ and likewise $\boxed{\mathbb{A}\mathbb{A}^{-1} = \mathbb{I}}$

⊛ Solving a system of linear Equations via the Inverse Matrix Method

- we saw $\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$ can be written

as a matrix $\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

or, simply $\boxed{\mathbb{A} \quad \vec{x} = \vec{b}}$

- If we multiply both sides by $\mathbb{A}^{-1}$, from the left we have

$$\mathbb{A}^{-1}\,\mathbb{A}\,\vec{x} = \mathbb{A}^{-1}\,\vec{b}$$

- But $\mathbb{A}^{-1}\,\mathbb{A} = \mathbb{I}$ ← identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\mathbb{I}\quad\vec{x} = \mathbb{A}^{-1}\,\vec{b}$$

- But $\mathbb{I}\,\vec{x} = \vec{x}$

So finally $\boxed{\vec{x} = \mathbb{A}^{-1}\,\vec{b}}$

We have solved for $\vec{x}$!

$\boxed{\text{Ex}}$ we saw in the Last Example that

$\mathbb{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ has an inverse $\mathbb{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

So the solution to the system: $\begin{cases} x = 4 \\ -x+y=-3 \end{cases}$ ⇔ $\underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\mathbb{A}}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 4 \\ -3 \end{pmatrix}}_{\vec{b}}$

is $\vec{x} = \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}^{\mathbb{A}^{-1}}\overbrace{\begin{pmatrix} 4 \\ -3 \end{pmatrix}}^{\vec{b}} = \begin{pmatrix} 4 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}}$

Ex

If $A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

and $A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -1/2 & 3/2 & -1 \end{bmatrix}$, as we will show soon,

Solve the system using A^{-1}:

$$\begin{aligned} 3x - 2y + 4z &= 1 \\ x \quad\quad + 2z &= 2 \\ y \quad\quad &= 3 \end{aligned}$$

- convert the system to matrices

$$\begin{pmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Then since this is $A\vec{x} = \vec{b}$ we see

$$\vec{x} = A^{-1}\vec{b} \quad \text{or}$$

OR $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -1/2 & 3/2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-4+6 \\ 2 \\ -\frac{1}{2}+3-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1/2 \end{pmatrix}$

So $\boxed{(x, y, z) = (3, 2, -1/2)}$

Finding A^{-1}

1×1: $[a]A^{-1} = [1] \rightarrow$ $\boxed{\boxed{A^{-1} = [1/a]}}$ if $A = [a]$ then

Test $[a][1/a] \stackrel{?}{=} [1]$ ✓

2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ← given, what is $A^{-1} = ?$

- let $A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$
- then since

$\mathbb{I} = A \cdot A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

we have

$$\left[\begin{array}{c|c} ae+bg & af+bh \\ \hline ce+dg & cf+dh \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- 4 eqns in $\underbrace{e, f, g, h}$

$ae + bg = 1$	$af + bh = 0$
$ce + dg = 0$	$cf + dh = 1$
$e = -\left(\frac{d}{c}\right) g$	$f = -\left(\frac{b}{a}\right) h$
$a\left(\frac{-d}{c}\right)g + bg = 1$	$c\left(\frac{-b}{a}\right)h + dh = 1$
$g = \frac{1}{-da/c + b} \cdot \frac{c}{c}$	$h = \frac{1}{-bc/a + d} \cdot \frac{a}{a}$ ⇨

⑥

- simplifying

$$g = \frac{c}{bc-da} \cdot \frac{-1}{-1} \qquad h = \frac{a}{ad-bc}$$

$$\boxed{g = \frac{-c}{ad-bc}} \qquad \boxed{h = \frac{a}{ad-bc}}$$

- we also need "e" $\longrightarrow$ and $d \longrightarrow$ "f" :

Use $e = -\frac{d}{c} \cdot g$ and $f = -\frac{b}{a} h$

$$e = -\frac{d}{c}\left[\frac{-c}{ad-bc}\right] \qquad f = -\frac{b}{a}\left[\frac{a}{ad-bc}\right]$$

$$\boxed{e = \frac{d}{ad-bc}} \qquad \boxed{f = \frac{-b}{ad-bc}}$$

- Form A^{-1}:

$$A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \frac{d}{D} & \frac{-b}{D} \\ -c/D & a/D \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{D}}$$

formula for 2x2 inverse matrix

where $D = ad - bc$

We call D the <u>determinant</u> of A.

Ex Find the inverse of $\mathbb{A} = \begin{bmatrix} 1 & -3 \\ 4 & 7 \end{bmatrix}$

$$\mathbb{A} = \begin{bmatrix} 1 & -3 \\ 4 & 7 \end{bmatrix}$$

a, b, c, d

$$\mathbb{A}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$$

- formula for 2x2 yeilds:

$$\mathbb{A}^{-1} = \frac{\begin{bmatrix} 7 & -(-3) \\ -4 & 1 \end{bmatrix}}{1\cdot 7 - 4\cdot(-3)}$$

- exchange the main diag'l
- change signs on anti-dig'l
- divide by the determinant

OR

$$\mathbb{A}^{-1} = \frac{\begin{bmatrix} 7 & 3 \\ -4 & 1 \end{bmatrix}}{19} = \boxed{\begin{bmatrix} 7/19 & 3/19 \\ -4/19 & 1/19 \end{bmatrix}}$$

- Test: $\mathbb{A}^{-1}\mathbb{A} \stackrel{?}{=} \mathbb{I}$

$$\frac{1}{19}\begin{bmatrix} 7 & 3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 4 & 7 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{19}\begin{bmatrix} 7\cdot 1 + 3\cdot 4 & 7(-3)+3(7) \\ -4\cdot 1 + 1\cdot 4 & -4(-3)+1(7) \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{19}\begin{bmatrix} 19 & 0 \\ 0 & 19 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yes!!

⊛ General procedures for matrix inversion

"Armani Method" Armani $A | X$ {any $n \times n$

- The Gaussian method of matrix inversion.

For 2×2 and higher dimensions follow these steps to produce an inverse (if it exists)

(i) Form the augmented matrix $[\mathbb{A} \mid \mathbb{I}]$

(ii) Perform Row ops until you transform $[\mathbb{A} \mid \mathbb{I}]$ into $[\mathbb{I} \mid \mathbb{B}]$ form

(iii) The $\mathbb{B}$ is $\mathbb{A}^{-1}$

Ex Find $\mathbb{A}^{-1}$ if $\mathbb{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

(i) $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right]$ $\times 1$

(ii) $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$

(iii) $\boxed{\mathbb{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$

Ex Find the inverse of $\mathbb{A} = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

(i) Form augmented matrix $[\mathbb{A} \mid \mathbb{I}]$

$$\left[\begin{array}{ccc|ccc} 3 & -2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} r_2 \to r_1 \\ r_1 \to r_3 \end{array}$$

(ii) Convert the LHS to $\mathbb{I}$ via Row Operations:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & -2 & 4 & 1 & 0 & 0 \end{array}\right] * -3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 1 & -3 & 0 \end{array}\right] * 2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -3 & 2 \end{array}\right] * 1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -3 & 2 \end{array}\right] \div -2$$

(iii)

$$\boxed{\mathbb{A}^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -1/2 & 3/2 & -1 \end{bmatrix}}$$

- check via matrixcalc.org OR multiply $\mathbb{A}\mathbb{A}^{-1} \stackrel{?}{=} \mathbb{I}$

Math 104 CW 13 Name ______

1. Use the "Armani Method" to find the inverse of $\mathbb{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

Math 104 CW Name

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EX Use the "Armani Method" (Gaussian Method) to find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

(i) form $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} *-2 \,;\, *-1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ *2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ *-3 \,;\, *3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} \\ *-2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \div -1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

So

$$\boxed{A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}}$$

Now lets solve a system!

Ex Solve

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2x + 5y + 3z &= -4 \\ x + 8z &= 2 \end{aligned}$$

(i) write in matrix form

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 11 \\ -4 \\ 2 \end{pmatrix}}_{\vec{b}}$$

inverse was found in the previous example

(ii) write in inverse form

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 11 \\ -4 \\ 2 \end{pmatrix}}_{\vec{b}}$$

(iii) perform the multiplication

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -40(11) + 16\cdot(-4) + 9(2) \\ 13(11) \quad -5(4) \quad -3(2) \\ 5(11) \quad -2(4) \quad -1(2) \end{pmatrix} = \begin{pmatrix} -486 \\ -15 \\ 45 \end{pmatrix}$$

(iv) So $\boxed{(x, y, z) = (-486, -15, 45)}$

please include this part in your answers. (thank you)

- The inverse method really only saves time if you need to solve $A\vec{x} = \vec{b}$ for multiple $\vec{b}$'s but same A.