9.4 Partial Fractions

- We learned to combine fractions that contain a variable (polynomial).

Ex

$$\frac{3}{x-1}+\frac{4}{x+2}$$

$$=\frac{3}{x-1}\frac{(x+2)}{(x+2)}+\frac{4}{x+2}\frac{(x-1)}{(x-1)}$$

$$=\frac{3x+6+4x-4}{(x-1)(x+2)}$$

$$=\frac{7x+2}{(x-1)(x+2)}$$

i.e. $\boxed{\frac{3}{x-1}+\frac{4}{x+2}=\frac{7x+2}{(x-1)(x+2)}}$

- In this section we desire to go in reverse.

i.e, given $\frac{7x+2}{(x-1)(x+2)}$ what are A & B

such that $\frac{7x+2}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$?

Procedures :

- Given a rational function
  - (i) Factor the denominator, ((Long divide until it is a proper rational function.))
  - (ii) Select the form that the function will take which posses numerators A, B, etc.
  - (iii) Apply one of the methods to determine the Numerators Constants.
  - (iv) state final answer.

EX Decompose $\dfrac{7x+2}{x^2+x-2}$:

(i) • note deg top < deg bot $\Rightarrow$ proper fraction

• factor the denominator

$$\frac{7x+2}{x^2+x-2}$$

$$= \frac{7x+2}{(x-1)(x+2)}$$

(ii) • form

$$\frac{7x+2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$\boxed{\text{EX}}$ cont.

(iii) <u>traditional method</u> used to determine unknown numerator values

- obtain a common denominator

$$\frac{7x+2}{(x-1)(x+2)} = \frac{A}{x-1}\left(\frac{x+2}{x+2}\right) + \frac{B}{x+2}\left(\frac{x-1}{x-1}\right)$$

$$\Rightarrow \frac{7x+2}{(x-1)(x+2)} = \frac{Ax+2A+Bx-B}{(x-1)(x+2)}$$

- equate the numerators

$$\boxed{7}x+\boxed{2} = \boxed{(A+B)}x + \boxed{(2A-B)}$$

- Using the Principle of Linear Indepence we cover in Linear Algebra we need the following to be met:

match powers

$$\begin{array}{lcl} & \text{LHS} & = \text{RHS} \\ x^1 : & 7 & = A+B \\ \oplus\ x^0 : & 2 & = 2A-B \end{array} \Bigg\} \quad 2\times 2 \text{ system}$$

- solve this 2x2: Elimination

$$9 = 3A \longrightarrow \boxed{A=3}$$

insert into x^1 eqn: $7 = 3+B \rightarrow \boxed{B=4}$

- answer

$$\boxed{\frac{7x+2}{(x-1)(x+2)} = \frac{3}{x-1} + \frac{4}{x+2}}$$

EX

Decompose

• form

$$\frac{4x^2-8x+7}{(x-2)(x+1)(x+3)} = \frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{x+3}$$

• common denominators

$$= \frac{A(x+1)(x+3)+B(x-2)(x+3)+C(x-2)(x+1)}{(x-2)(x+1)(x+3)}$$

• equate numerators:

$$4x^2-8x+7 = Ax^2+4Ax+3A+Bx^2+Bx-6B+Cx^2-Cx-2C$$

• equate powers of x:

$$\left\{\begin{array}{l} x^2:\ 4 = A+B+C \\ x^1:\ -8 = 4A+B-C \\ x^0:\ 7 = 3A-6B-2C \end{array}\right\} 3\times 3 \text{ (see sec 9.2)}$$

Next Page for {Details by hand} ⇒

• solve

-or- use matrixcalc.org $\Rightarrow A = \frac{7}{15},\ B = \frac{-19}{6},\ C = \frac{67}{10}$

• answer

$$\Rightarrow \boxed{\frac{4x^2-8x+7}{(x-2)(x+1)(x+3)} = \frac{7}{15(x-2)} - \frac{19}{6(x+1)} + \frac{67}{10(x+3)}}$$

• Solve

$$\begin{aligned} A + B + C &= 4 \quad *-4\ ;\ *-3 \\ 4A + B - C &= -8 \\ 3A - 6B - 2C &= 7 \end{aligned}$$

//

$$\begin{aligned} A + B + C &= 4 \\ -3B - 5C &= -24 \quad *-3 \\ -9B - 5C &= -5 \end{aligned}$$

//

$$\begin{aligned} A + B + C &= 4 \\ 0 \quad 3B + 5C &= 24 \\ 0 \quad 0 \quad 10C &= 67 \end{aligned}$$

$A = \frac{7}{15}$

$B = -\frac{19}{6}$

$C = \frac{67}{10}$

{check @ matrixcalc.org}

• Thus

Answer

$$\boxed{\frac{4x^2-8x+7}{(x-2)(x+1)(x+3)} = \frac{7}{15(x-2)} - \frac{19}{6(x+1)} + \frac{67}{10(x+3)}}$$

• Why do we do this? multiple applications

In Calculus we are asked to integrate:

$$\int \frac{4x^2-8x+7}{(x-2)(x+1)(x+3)}\,dx = \frac{7}{15}\int \frac{dx}{x-2} - \frac{19}{6}\int \frac{dx}{x+1} + \frac{67}{10}\int \frac{dx}{x+3}$$

$$= \frac{7}{15}\ln(x-2) - \frac{19}{6}\ln(x+1) + \frac{67}{10}\ln(x+3)$$

Higher Multiplicity Fractions

Ex $\dfrac{-5x-19}{(x+4)^2} = \dfrac{A}{x+4} + \dfrac{B}{(x+4)^2}$

For higher powers on denominator terms we need to include all lessors powers.

Ex $\dfrac{2x^2+4x-1}{(x-3)^3} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{C}{(x-3)^3}$ • form

• common denom

$$= \frac{A}{x-3}\left(\frac{x-3}{x-3}\right)^2 + \frac{B}{(x-3)^2}\left(\frac{x-3}{x-3}\right) + \frac{C}{(x-3)^3}$$

$$\frac{2x^2+4x-1}{(x-3)^3} = \frac{A(x^2-6x+9) + Bx-3B+C}{(x-3)^3}$$

• equate Numerators

$$2x^2+4x-1 = Ax^2 + (-6A+B)x + (9A-3B+C)$$

+

• equate powers:

x^2: $2 = A$ → $\underline{A=2}$

x^1: $4 = -6A+B$ → $\underline{B=16}$

x^0: $-1 = 9A-3B+C$ → $-1 = 9(2)-3(16)+C$

• solve

$-1 = 18-48+C$

$-1+30 = C$ $\quad \underline{C=29}$

• ANS:

$$\boxed{\frac{2x^2+4x-1}{(x-3)^3} = \frac{2}{x-3} + \frac{16}{(x-3)^2} + \frac{29}{(x-3)^3}}$$ ✓

⊛ Quadratic Fractions

Ex $\dfrac{2x+9}{x^2+4x+1}$ ← cannot factor "prime"

The fund. thm of algebra says that all polynomials factors into products of $(ax+b)$ type factors and (ax^2+bx+c) type.

linear

quadratic

{ cubic-like: ax^3+bx^2+cx+d will factor into a linear or quadratic }

• Form will need an "x" on top:

Ex (i) form

$$\frac{2x+9}{(x^2+4x+1)(x-1)} = \frac{Ax+B}{x^2+4x+1} + \frac{C}{x-1}$$

(ii) denom

$$\frac{2x+9}{(x^2+4x+1)(x-1)} = \frac{(Ax+B)(x-1) + C(x^2+4x+1)}{(x^2+4x+1)(x-1)}$$

(iii)

• equate numerators

$$\Rightarrow 0x^2 + 2x + 9 = Ax^2 + Bx - Ax - B + Cx^2 + 4Cx + C$$

(iv) match powers

x^2: $0 = A + C$

x^1: $2 = B - A + 4C$

x^0: $9 = -B + C$

(v) Solve

$\to A = -C$ or $-A = C$

$\Rightarrow 2 = B - A - 4A$, $\boxed{B - 5A = 2}$

$\oplus$ $\boxed{-B - A = 9}$

$-6A = 11$

$\boxed{A = -11/6}$

$\boxed{C = 11/6}$

$B = -\frac{54}{6} - \left(-\frac{11}{6}\right) = 3$

(vi) final

$$\boxed{\frac{2x+9}{(x^2+4x+1)(x-1)} = \frac{-\frac{11}{6}x - \frac{43}{6}}{x^2+4x+1} + \frac{\frac{11}{6}}{x-1}}$$

* quadratic with higher multiplicity

EX
$$\frac{x^3-5x^2+12x+144}{(x^2+3)(x^2-2x+2)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2-2x+2}$$

But if we need $(x^2+3)^2$ on bottom:

EX
$$\frac{x^3-5x^2+12x+144}{(x^2+3)^2(x^2-2x+2)} = \frac{?}{x^2+3} + \frac{?}{(x^2+3)^2} + \frac{?}{x^2-2x+2}$$

Now since we have quadratic denominators we need Linear numerators.

$$\frac{x^3-5x^2+12x+144}{(x^2+3)^2(x^2-2x+2)} = \boxed{\frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}} + \boxed{\frac{Ex+F}{x^2-2x+2}}$$

We then follow the steps in the usual way

Improper fractions

Ex

$$\frac{x^4-3x^2+x-1}{x^4+12x^3+36x^2} = \text{Form ?}$$

Step (0) we have deg on top = deg on bot

$\Rightarrow$ Long divide

$$\begin{array}{r|l} & 1 \\ x^4+12x^3+36x^2+0x+0 & x^4+0x^3-3x^2+x-1 \\ & -(x^4+12x^3+36x^2+0x+0) \\ \hline & -12x^3+33x^2+x-1 \end{array}$$

$\leftarrow$ remainder

Our fraction looks like:

$$\boxed{\frac{x^4-3x^2+x-1}{x^4+12x^3+36x^2} = 1 + \frac{-12x^3+33x^2+x-1}{x^4+12x^3+36x^2}}$$

Now we work on decomposing the 2^{nd} RHS term.

- factor denom:

$$\frac{-12x^3+33x^2+x-1}{x^4+12x^3+36x^2} =$$

$$= \frac{-12x^3+33x^2+x-1}{x^2(x^2+12x+36)}$$

$$= \frac{-12x^3+33x^2+x-1}{x^2(x+6)^2}$$

add the "1 +" when all done

9.4 is done

- Form

$$\frac{-12x^3+33x^2+x-1}{x^2(x+6)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+6} + \frac{D}{(x+6)^2}$$

...