7.5 Solving Trig Eqns.

⊛ Review Identities

- **Basic Def:** $\csc x = \dfrac{1}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$, $\cot x = \dfrac{1}{\tan x}$

  $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$

- **Pythagorean Identities:**

$$\boxed{\begin{aligned} &\cos^2 x + \sin^2 x = 1 \\ &\cot^2 x + 1 = \csc^2 x \\ &1 + \tan^2 x = \sec^2 x \end{aligned}}$$

- **Sum-to-product Identities:**

$$\boxed{\begin{aligned} &\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \\ &\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \end{aligned}}$$

- **half-angle:**

$$\boxed{\begin{aligned} &\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}} \Leftrightarrow \sin^2(x) = \frac{1}{2}(1-\cos(2x)) \\ &\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}} \Leftrightarrow \cos^2(x) = \frac{1}{2}(1+\cos(2x)) \end{aligned}}$$

- **Double-angle:**

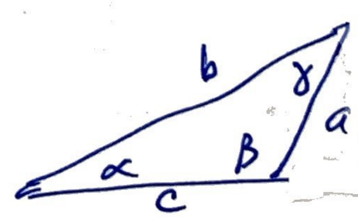
$$\boxed{\begin{aligned} &\sin(2x) = 2\cos(x)\sin(x) \\ &\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1 \end{aligned}}$$

- **Cofunction Identities:** (phase-shift):

$$\boxed{\begin{aligned} &\sin(x \pm \pi/2) = \pm\cos(x) \\ &\cos(x \pm \pi/2) = \mp\sin(x) \end{aligned}}$$

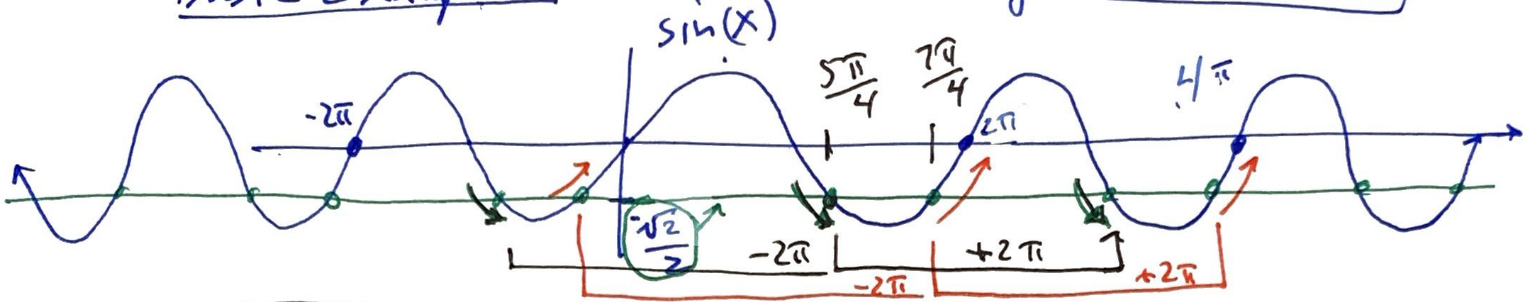
- : $\sin\alpha = \sin = \sin\gamma$

  $^2 = b^2 + c^2 - 2bc\cos$



(*) we seek to "solve for x" but not for exponential nor logs nor polynomials but rather for trig functions.

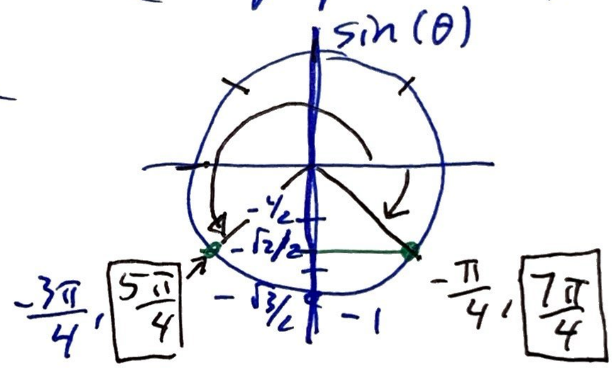
- Basic Examples look like $\boxed{\text{trig}(x) = \text{constant}}$



Ex solve for θ : $2\sin\theta = -\sqrt{2}$

$\Rightarrow \quad \sin\theta = -\sqrt{2}/2$ (see graph above)

use the trig circle



- family one: $\boxed{\frac{5\pi}{4} \pm 2\pi n}$ $n \in \mathbb{Z}$ Family of integers ... -3, -2, -1, 0, 1, 2, 3 ...
- family two: $\boxed{\frac{7\pi}{4} \pm 2\pi n}$ $n \in \mathbb{Z}$

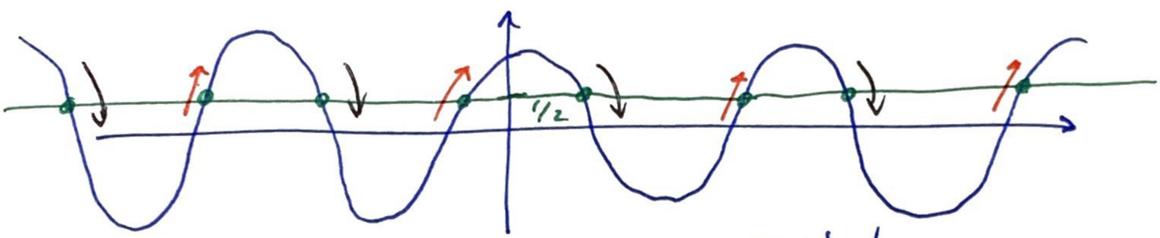
Q: between $x = [-2\pi, 4\pi]$ solutions are:

$$\left\{ -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4} \right\}$$

1. write the families of solutions for

$$\cos(\theta) = 1/2$$

(i)



period

$$\boxed{\begin{aligned}\theta &= \frac{\pi}{3} \pm 2\pi k \\ \theta &= \frac{5\pi}{3} \pm 2\pi k\end{aligned}}$$

any integer

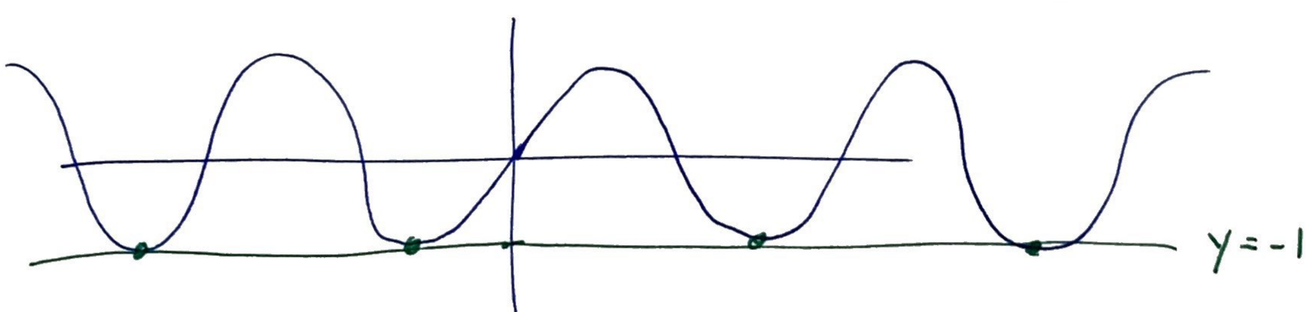
principle angle

2. Write the solutions for #1 in between

$[-2\pi, 3\pi] \xrightarrow{i.e.} \left[-\frac{6\pi}{3}, \frac{9\pi}{3}\right]$

- $\frac{\pi}{3} \pm \frac{6\pi}{3}k = \frac{\pi+6\pi}{3}, \cancel{\frac{\pi+12\pi}{3}}, \cancel{\frac{\pi+18\pi}{3}} \dots$ or $\frac{\pi-6\pi}{3}, \cancel{\frac{\pi-12\pi}{3}}, \dots$
- $\frac{5\pi}{3} \pm \frac{6\pi}{3}k = \cancel{\frac{5\pi+6\pi}{3}}, \cancel{\frac{5\pi+12\pi}{3}}, \dots$ or $\frac{5\pi-6\pi}{3}, \frac{5\pi-12\pi}{3}, \dots$

3 write solutions for $\sin(\theta) = -1$



- one family only: $\boxed{\frac{3\pi}{2} \pm 2\pi k}$

EX Solve for x

$$4\sin^2(x) - 2 = 0$$

• isolate $\sin(x)$

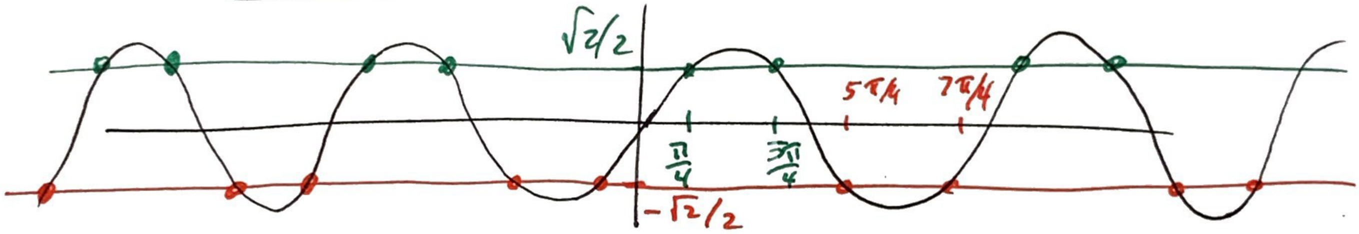
$$\sin^2(x) = 2/4 \Rightarrow \sin^2(x) = 1/2$$

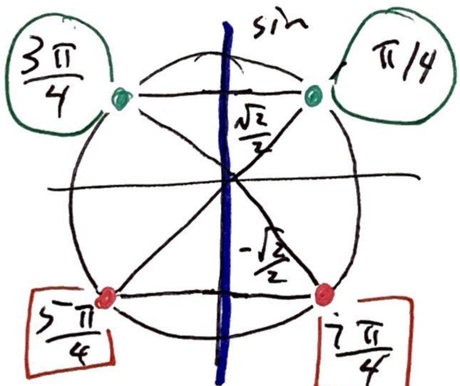
$$\Rightarrow \sin(x) = \pm\sqrt{1/2}$$

$$\boxed{\sin(x) = \pm\frac{\sqrt{2}}{2}}$$

(+) (−)

• Solve $\boxed{\sin(x) = \frac{\sqrt{2}}{2}}$ and $\boxed{\sin(x) = \frac{-\sqrt{2}}{2}}$





family:
1 : $\frac{\pi}{4} \pm 2\pi k$
2 : $\frac{3\pi}{4} \pm 2\pi k$
3 : $\frac{5\pi}{4} \pm 2\pi k$
4 : $\frac{7\pi}{4} \pm 2\pi k$

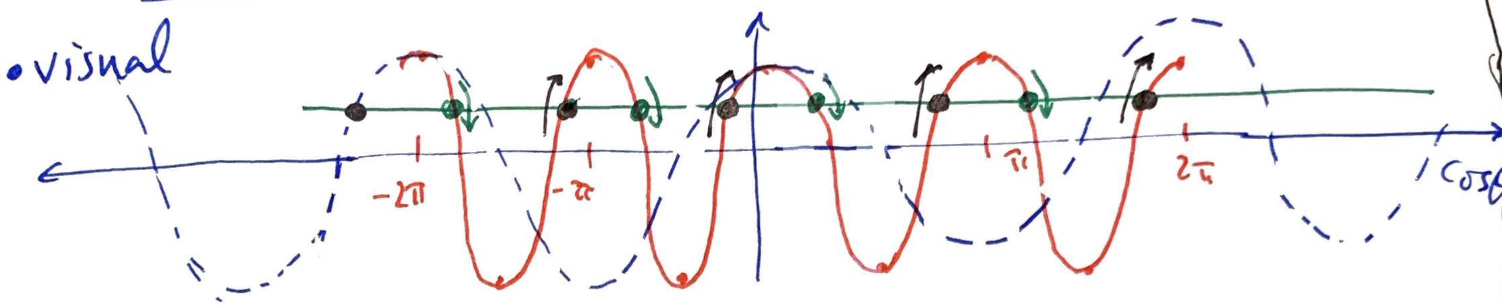
• Between $[0, 2\pi)$

$$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$

Ex ⊛ Change the period of the trig function

$\cos(2\theta) = 1/2$

• visual



$\cos(2\theta)$ $\quad P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

• analytical

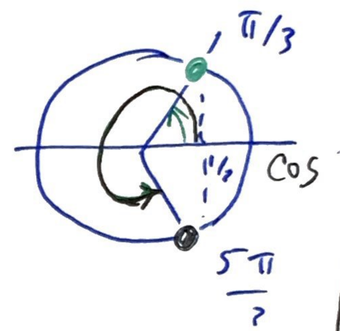
$2\theta = \frac{\pi}{3} \pm 2\pi k$ downward family

$2\theta = \frac{5\pi}{3} \pm 2\pi k$ upward family

÷ 2

$\theta_1 = \frac{\pi}{6} \pm \pi k$

$\theta_2 = \frac{5\pi}{6} \pm \pi k$



Ex Solve for x : $\csc^2(x) - 4 = 0$

(i) isolate the trig function

$$\csc(x) = \pm\sqrt{4}$$

$$\boxed{\csc(x) = \pm 2}$$

(ii) write using familiar trig if possible:

$$\frac{1}{\sin(x)} = \pm 2$$ reciprocate

$$\boxed{\sin(x) = \pm 1/2}$$

⊕ $\boxed{\sin(x) = \frac{1}{2}}$ ...

⊖ $\boxed{\sin(x) = -\frac{1}{2}}$

solve as done previously ...

EX $\sec(x)\sin(x) - 2\sin(x) = 0$

try factoring

$\Rightarrow [\sin(x)][\sec(x) - 2] = 0$

so $\boxed{\sin(x) = 0}$ or $\boxed{\sec(x) = 2}$ reciprocate

$\boxed{\cos(x) = 1/2}$

$\boxed{x = \pm n\pi}$

$$\boxed{\begin{aligned} x &= \pi/3 \pm 2\pi k \\ x &= \frac{5\pi}{3} \pm 2\pi k \end{aligned}}$$

EX $2\cos^2(t) + \cos(t) = 1$

quadratic in $\cos(t)$

letting $\boxed{u = \cos(t)}$

$2u^2 + u - 1 = 0$

factor

$(2u - 1)(u + 1) = 0$

- $2u - 1 = 0$

or

- $u + 1 = 0$

$2\cos(t) = 1$

$\boxed{\cos(t) = 1/2}$

$$\boxed{\begin{aligned} t_1 &= \frac{\pi}{3} \pm 2\pi k \\ t_2 &= \frac{5\pi}{3} \pm 2\pi k \end{aligned}}$$

$\cos(t) = -1$

$\boxed{\cos(t) = -1}$

$\boxed{t_3 = \pi \pm 2\pi k}$

EX Solve for x: $\boxed{\sin(3x)\cos(6x) - \cos(3x)\sin(6x)} = -0.9$

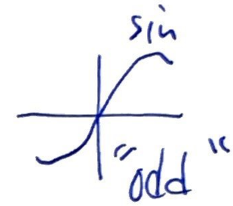
- this looks like a sum-to-product rule (review formulas)

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

use this one $\rightarrow$ $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

$A \rightarrow 3x$, $B \rightarrow 6x$

$$\boxed{\sin(3x - 6x)} = -0.9$$



$$\sin(-3x) = -0.9$$

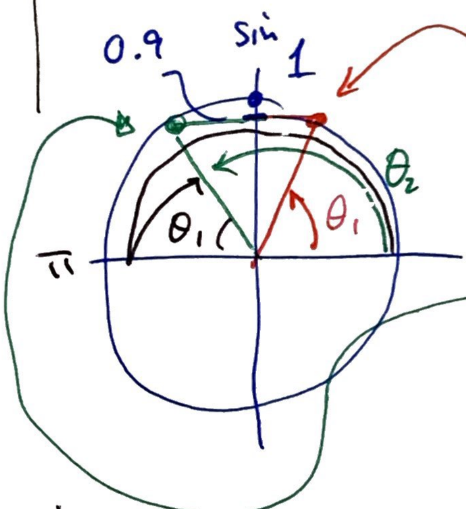
$$-\sin(3x) = -0.9$$

$$\boxed{\sin(3x) = 0.9} \text{ solve}$$

- No "pretty numbers"

$$\boxed{\theta_1 = \frac{\sin^{-1}(0.9)}{3}}$$

$\sin^{-1}(0.9) = 1.1198$



$$3\theta_2 = \pi - \theta_1$$

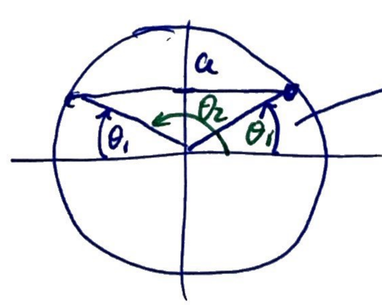
$$\boxed{\theta_2 = \frac{\pi}{3} - \frac{\sin^{-1}(0.9)}{3}}$$

[0 to π]

⊕ In general $\sin(x) = a$

$$\theta_2 = \pi - \theta_1$$

$$\boxed{\theta_2 = \pi - \sin^{-1}(a)}$$



$$\boxed{\theta_1 = \sin^{-1}(a)}$$

similar for $\cos(x) = a$

Ex $\cos(6x) - \cos(3x) = 0$

- Numerically go to desmos

$$\cos(6x) = \cos(3x)$$

From Desmos we see that there are the $\pm\frac{2\pi}{3}$ families

- we can analytically justify the $\pm\frac{2\pi}{3}$ family

$$6x \pm 2\pi k = 3x \pm 2\pi l$$

$$3x = \pm 2\pi l \mp 2\pi k$$

$$x = \pm\left(\frac{2\pi}{3} l \mp \frac{2\pi}{3} k\right)$$

$$\boxed{x = \pm\frac{2\pi}{3}(\underbrace{l \mp k}_{n})}$$

$$\boxed{x = \pm\frac{2\pi}{3} n}$$ done? No!

But we also see from Desmos that there is a $\frac{2\pi}{9}$ family and a $\frac{4\pi}{9}$ family

- So where do we get these other families from?
- Note that $\cos(2\theta) = 2\cos^2(\theta) - 1$

then $\cos(6x) - \cos(3x) = 0$

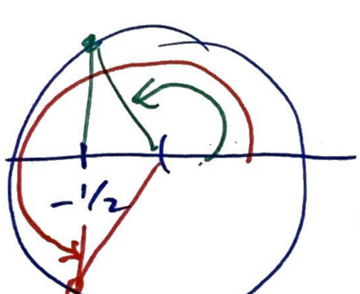
becomes $2\cos^2(3x) - 1 - \cos(3x) = 0$

or $\boxed{2\cos^2(3x) - \cos(3x) - 1 = 0}$ quadratic like function

factors

$$(2\cos(3x) + 1)(\cos(3x) - 1) = 0$$

- $\boxed{\cos(3x) = -\frac{1}{2}}$



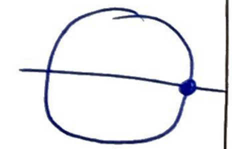
- $3x = \frac{2\pi}{3} + 2\pi n$

$$\boxed{x = \frac{2\pi}{9} + \frac{2\pi}{3} n}$$

- $3x = \frac{4\pi}{3} + 2\pi n$

$$\boxed{x = \frac{4\pi}{9} + \frac{2\pi}{3} n}$$

- $\boxed{\cos(3x) = 1}$



$$3x = \pm 2\pi n$$

$$\boxed{x = \pm\frac{2\pi}{3} n}$$ The original family

Moral of the story...
avoid cancelling across, or moving terms across, the equal sign.

EX Solve both numerically & analytically

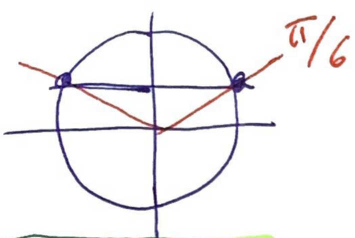
$$6\sin^2 x - 5\sin x + 1 = 0$$

Numerical Approach

$$\boxed{6\sin^2(x) = 5\sin(x) - 1}$$

- factor $(2\sin(x) - 1)(3\sin(x) - 1) = 0$

$\sin(x) = \frac{1}{2}$

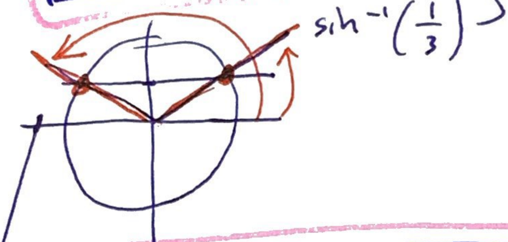


$$\boxed{x = \frac{\pi}{6} \pm 2\pi n}$$

$$\boxed{x = \frac{5\pi}{6} \pm 2\pi n}$$

$\sin(x) = \frac{1}{3}$

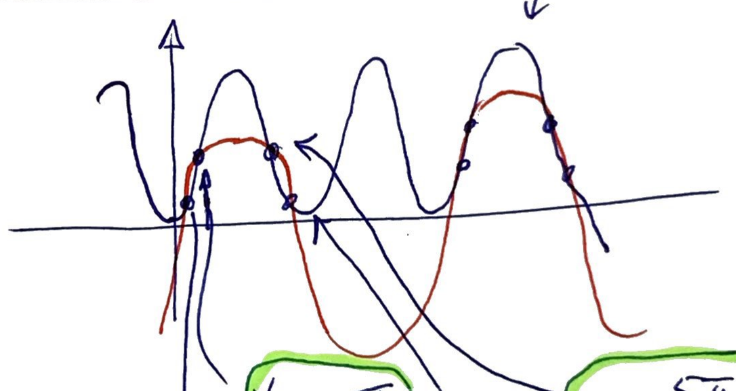
$$\boxed{x = \sin^{-1}\left(\tfrac{1}{3}\right) \pm 2\pi n}$$



$$\boxed{x = \pi - \sin^{-1}\left(\tfrac{1}{3}\right) \pm 2\pi n}$$

- On Desmos:

LHS:	$y = 6\sin^2(x)$
RHS:	$y = 5\sin(x) - 1$



$\boxed{x = \frac{\pi}{6}}$ $\quad$ $\boxed{x = \frac{5\pi}{6}}$

$x = 0.3398$ $\quad$ $x = 2.802$

Compare to analytical:

Note that
$$\begin{cases} \sin^{-1}(0.3333) = 0.3398 \\ \text{and} \quad \pi - \sin^{-1}(0.3333) = 2.8017 \end{cases}$$