4.5 Properties of Logarithms

⊛ Equivalent Problem properties

$$a^x = b \iff \log_a(b) = x$$

- $a^0 = 1 \iff \log_a(1) = 0$
- $a^1 = a \iff \log_a(a) = 1$
- $a^2 = a^2 \iff \log_a(a^2) = 2$
- $a^r = a^r \iff \log_a(a^r) = r$

⊛ Inverse function / Function properties

let $f(x) = a^x$, then $f^{-1}(x) = \log_a(x)$

So $(f \circ f^{-1})(x) = x$

$f(f^{-1}(x)) = x$

$$a^{\log_a(x)} = x$$

- likewise

$(f^{-1} \circ f)(x) = x$

$$\log_a(a^x) = x$$

(*) General properties

- $a^{n+m} = a^n a^m \leftrightarrow \log_a(a^n a^m) = n+m$

- let $a^n = N \leftrightarrow \log_a(N) = n$
- let $a^m = M \leftrightarrow \log_a(M) = m$

$$\log_a(NM) = \log_a(N) + \log_a(M)$$

{Note $\log_a(N+M) \neq \log_a(N) + \log_a(M)$
$\Rightarrow$ the $\log_a$ functions are non-linear}

- Next

$$a^{n-m} = \frac{a^n}{a^m} \leftrightarrow \log_a\left(\frac{a^n}{a^m}\right) = n-m$$

$$\log_a\left(\frac{N}{M}\right) = \log_a(N) - \log_a(M)$$

- Next

$$(a^m)^r = a^{mr} \leftrightarrow \log_a(a^{mr}) = mr$$

$$\log_a(M^r) = r \log_a(M)$$

⊗ change of base

$$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$$

- derivation

⊛ Applications of the properties

To solve logarithmic eqns we seek to

Combine log terms

Ex Combine into one logarithm

$$\log(7) + \log(x) + \log(y)$$

$$= \boxed{\log(7 \cdot x \cdot y)}$$

Ex

$$3\ln(x) - 2\ln(x-1) + \ln(5)$$

$$= 3\ln(x) + \ln(5) - 2\ln(x-1)$$

$$= \ln(x^3) + \ln(5) - \ln(x-1)^2$$

$$= \ln(x^3 \cdot 5) - \ln(x-1)^2$$

$$= \boxed{\ln\left(\frac{5x^3}{(x-1)^2}\right)}$$

⑧ Expand a single logarithm into sums & differences

Ex $\log\left(\sqrt{x^3 y^{-4}}\right)$

$\sqrt{a} = a^{1/2}$

$= \log\left((x^3 y^{-4})^{1/2}\right)$

$= \frac{1}{2}\log\left(x^3 y^{-4}\right)$

$= \frac{1}{2}\left[\log(x^3) + \log(y^{-4})\right]$

$= \frac{1}{2}\left[3\log(x) + (-4)\log(y)\right]$

$= \boxed{\frac{3}{2}\log(x) - 2\log(y)}$

✱ Change of Base application

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

EX Use a calculator an evaluate

$$\log_3(18)$$

$$= \frac{\ln(18)}{\ln(3)}$$

18 [LN] [÷] 3 [LN] [=]

$$= \frac{2.89037}{1.09861}$$

$$= \boxed{2.63093}$$

wait

$$\log_3(18)$$

$$= \frac{\log(18)}{\log(3)}$$

18 [LOG] [÷] 3 [LOG] [=]

$$= \boxed{2.63093}$$ the same.