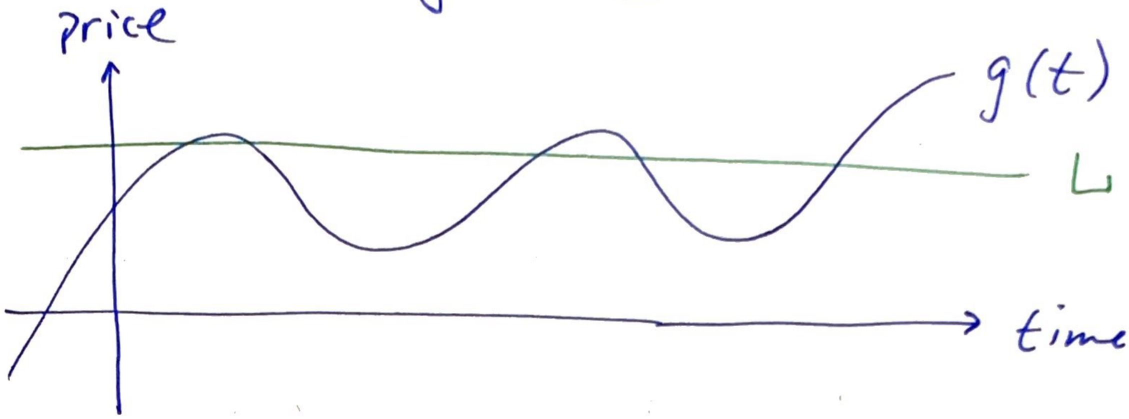
3.6 Zeros of Polynomials



let, $S(t) = g(t) - L$

Sell the position when $g(t) = L$

i.e. when $\underline{S(t) = 0}$

There are many models that use polynomials and most of the time people what to know where the polynomial has zeros

- Objective in this section is to discover the zeros (roots) of $f(x)$. ie $\boxed{f(x)=0 \text{ where?}}$

How: we use thms to guide us in locating these zeros; quantity, signs & candidates.

* Tools we need

Just as $\frac{13}{2} = 6 + 1 \Rightarrow 6 + \frac{1}{2}$ or $6\frac{1}{2}$

Likewise $\boxed{\frac{f(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}} \leftarrow 3.5$

multiply by $q(x)$

$$\boxed{f(x) = g(x)\,q(x) + r(x)}$$

- let $q(x) = x - d$
- then $\boxed{f(x) = (x-d)\,g(x) + r}$

the $g(x)$ is the bottom line of the synthetic division of $x-d$ into $f(x)$

$$\begin{array}{r|l} -(-d) & f(x) \\ \hline & g(x) \quad + r \end{array}$$

Ex $\frac{x^3 + 2x + 1}{x + 1}$

$$\begin{array}{r|rrrr} & x^3 & x^2 & x^1 & x^0 \\ -1 & 1 & 0 & 2 & 1 \\ & & -1 & 1 & -3 \\ \hline & 1 & -1 & 3 & -2 \\ & x^2 & x^1 & x^0 & r \end{array}$$

reduced polynomial $x^2 - x + 3$

$$\frac{x^3+2x+1}{x+1} = (1 \cdot x^2 - 1 \cdot x + 3) + \frac{-2}{x+1} \quad * (x+1)$$

$$x^3 + 2x + 1 = (x^2 - x + 3)(x+1) - 2$$

So $f(x) = (x-d)g(x) + r$ form

1. If d is a zero of $f(x)$ then $r=0$
   and $f(x) = (x-d)g(x)$ { partially factored $f(x)$ }

2. $f(d) = r$

- Next assume $x-d$ is a factor of $f(x)$

  { so $f(d) = 0$ }

  and $\boxed{f(x) = (x-d)g(x)}$

we then factor $g(x)$ to get $g(x) = (x-e)h(x)$

So $f(x) = (x-d)(x-e)h(x)$

we then factor $h(x)$ to get $h(x) = (x-f)j(x)$

So $f(x) = (x-d)(x-e)(x-f)j(x)$

we continue this process until $f(x)$ is

fully factored

$$f(x) = c\underbrace{(x-d)(x-e)(x-f)\cdots(x-p)}_{n \text{ linear factors.}}$$

when $f(x)=0$ is desired, then

$d, e, f, \ldots, p$ are the zeros!!

⊛ Quantify

Q: How many ~~zero~~ does a polynomial have?

A: "n" where the power of the highest term is "n" {Fund. Thm of Algebra}

Q: How many of these zeros are positive?

A: DeCartes rule of signs

Thm: Let $f(x)$ be a polynomial in std. form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0$$

- then the number of sign changes in the coefficents of $f(x)$ is equal to the number of positive zeros, less an even #
- Also the number of sign changes in the coefficients of $f(-x)$ is equal to the number of negative zeros, less an even #

EX let $f(x) = 4x^4 - 2x^2 + 5x + 1$ $\Rightarrow$ $\boxed{2 \text{ or } 0 \ (+) \text{ roots}}$

then $f(-x) = 4x^4 - 2x^2 - 5x + 1$ $\Rightarrow$ $\boxed{2 \text{ or } 0 \ (-) \text{ roots}}$

⊛ Candidates

Thm "Rational Zeros Theorem"

Let $f(x) = a_n x^n + \cdots a_1 x + a_0$

If the polynomial has rational zeros, of the form $\boxed{p/q}$, then p is a factor of a_0 and q is a factor of a_n

- it may be that there are no rational zeros i.e. complex conjugates, irrational conjugates.
- we write the pool of candidates as

$$\boxed{\pm \frac{\text{factors of } a_0}{\text{factors of } a_n}}$$

EX write the candidates for rational zeros of $f(x) = 4x^4 - 2x^2 + 5x + 3$

$a_0 = 3 \rightarrow$ factors of $\underline{3, 1}$

$a_n = 4 \rightarrow$ factors of $\underline{4, 2, 1}$

so possible rational zero candidates are

$$\pm \frac{3, 1}{4, 2, 1} = \pm \left(\frac{3}{4}, \frac{3}{2}, \frac{3}{1}, \frac{1}{4}, \frac{1}{2}, \frac{1}{1} \right)$$

In increasing order of possible rational zeros of $f(x)$:

$$\boxed{-3, -\frac{3}{2}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 3}$$

① Bounds on the zeros

- If the number "u" divided into the polynomial via synthetic division yeild a product line (bottom) of positive numbers we need not search for zeros greater than "u" Upper-Bound

Ex

$$\begin{array}{r|rrrr} 2 & 3 & 11 & -6 & -8 \\ & & 6 & 34 & 56 \\ \hline & 3 & 17 & 28 & 48 \end{array}$$

← all (+) numbers

So there is no rational zero above 2.

- If the number "l" is divided into the polynml via synthetic division yeilds a bottom line of alternation sign, "l" is a lower bound for the rational zero search.

Ex

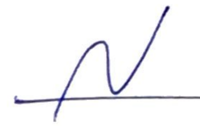
$$\begin{array}{r|rrrr} -8 & 3 & 11 & -6 & -8 \\ & & -24 & 104 & -784 \\ \hline & 3 & -13 & 98 & -792 \end{array}$$

← alternating numbes

So we need not search for zeros less than -8

EX "Seeking Zeros"

Factor $p(x) = 3x^3 + 11x^2 - 6x - 8$



1. quantity: degree 3 so $\boxed{\text{3 zeros}}$

2. number of (+) zeros:

$$p(x) = 3x^3 \underbrace{+ 11x^2 -}_{1} 6x - 8 \qquad \boxed{1\ (+)}$$

3. number of (−) zeros:

$$p(-x) = \underbrace{-3x^3 +}_{1} 11x^2 \underbrace{+ 6x -}_{2} 8 \qquad \boxed{2 \text{ or } 0}$$

4. Candidate List for rational zeros

$a_0 = 8 \rightarrow 8, 4, 2, 1$

$a_3 = 3 \rightarrow 3, 1$

$$p/q = \pm \frac{8,4,2,1}{3,1} = \pm\left(\frac{8}{3}, \frac{4}{3}, \frac{2}{3}, \frac{1}{3}, \frac{8}{1}, \frac{4}{1}, \frac{2}{1}, \frac{1}{1}\right)$$

ordered list:

$$\boxed{\left\{-8, -4, -\frac{8}{3}, -2, -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, 2, \frac{8}{3}, 4, 8\right\}}$$

5. Start the search. Be mindful of bounds

• try "2"

$$\begin{array}{r|rrrr} 2 & 3 & 11 & -6 & -8 \\ & & 6 & 34 & 56 \\ \hline & 3 & 17 & 28 & 48 \end{array}$$

← all (+) numbers

$\neq 0$ So 2 is an upper limit.

• try "−2"

$$\begin{array}{r|rrrr} -2 & 3 & 11 & -6 & -8 \\ & & -6 & -10 & 32 \\ \hline & 3 & 5 & -16 & 24 \end{array}$$

$\neq 0$

• try "−4"

$$\begin{array}{r|rrrr} -4 & \overset{x^3}{3} & 11 & -6 & -8 \\ & & -12 & 4 & 8 \\ \hline & \underset{x^2}{3} & -1 & -2 & \boxed{0} \end{array}$$

a zero!

opposite

So $p(x) = (x+4)(3x^2-x-2)$

6. Start all over: Find the zeros $g(x) = 3x^2-x-2$

Since this is quadratic use the formula

$$x = \frac{-(-1) \pm \sqrt{(-1)^2-4(3)(-2)}}{2(3)}$$

$$= \frac{1}{6} \pm \frac{\sqrt{25}}{6} \rightarrow \frac{1}{6}+\frac{5}{6},\ \frac{1}{6}-\frac{5}{6} \rightarrow \boxed{1, -\frac{2}{3}}$$

$$\boxed{p(x) = (x+4)(x-1)(x+\tfrac{2}{3})}$$

Zeros are $-4, 1, -\frac{2}{3}$ $\boxed{-4, -\frac{2}{3}, 1}$

EX Fully factor $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

1. $N = 4$ so 4 zeros expected

2. num(+) : $\boxed{\text{2 or 0 pos. zeros}}$

3. num(-) : $f(-x) = x^4 - 2x^3 - 9x^2 + 2x + 8$

$\boxed{\text{2 or 0 neg. zeros}}$

4. Candidates : $\left.\begin{array}{l} a_0 = 8, 4, 2, 1 \\ a_4 = 1 \end{array}\right\} \pm \dfrac{8, 4, 2, 1}{1}$

List $\boxed{\{-8, -4, -2, -1, 1, 2, 4, 8\}}$

5. search :
try 1

$$\begin{array}{r|rrrrr} & x^4 & x^3 & x^2 & x^1 & x^0 \\ 1 & 1 & 2 & -9 & -2 & 8 \\ & & 1 & 3 & -6 & -8 \\ \hline & 1 & 3 & -6 & -8 & \boxed{0} \\ & x^3 & x^2 & x^1 & x^0 & \end{array}$$

$x = 1$ is a zero since remainder $= 0$

So $\boxed{f(x) = (x-1)(x^3 + 3x^2 - 6x - 8)}$

6. Start over ... but $g(x)$ has the same $\begin{array}{l} a_3 = 1 \\ a_0 = 8 \end{array}$
So keep the same list.

EX (cont.)

6. (cont.)

try $x = 1$ again

$$\begin{array}{r|rrrr} & \multicolumn{4}{c}{\overbrace{\hspace{8em}}^{g(x)}} \\ 1 & 1 & 3 & -6 & -8 \\ & & 1 & 4 & -2 \\ \hline & 1 & 4 & -2 & -10 \end{array} \quad \times$$

try $x = -1$

$$\begin{array}{r|rrrr} & x^3 & x^2 & x^1 & x^0 \\ -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \\ & x^2 & x^1 & x^0 & \end{array}$$

$$f(x) = (x-1)(x+1)\underbrace{(x^2+2x-8)}_{(x-2)(x+4)}$$

factor!

$$\boxed{f(x) = (x-1)(x+1)(x-2)(x+4)}$$

← line-like since multiplicity $= 1$

$\boxed{\text{zeros are } 1, -1, 2, -4}$

- Sketch it: