11.4 Summation (Series)

- We now add the members of the sequence together:

Ex Write the summation of the sequence

$\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \dots\}$

$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \dots$

- The sequence with the general term a_n has a series $\boxed{S = \sum_{n=1}^{\infty} a_n}$

$$\boxed{S = a_1 + a_2 + a_3 + \dots} \Big\} \ \infty\text{-sum}$$

⊛ Partial sum ... terminate at N

Ex Write S_5 if $a_n = \frac{1}{n+1}$

$$S_5 = \underset{n=1}{\frac{1}{1+1}} + \underset{n=2}{\frac{1}{2+1}} + \underset{n=3}{\frac{1}{3+1}} + \underset{n=4}{\frac{1}{4+1}} + \underset{n=5}{\frac{1}{5+1}}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \qquad LCM = 60$$

$$= \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} + \frac{10}{60}$$

$$= \boxed{\frac{87}{60}}$$

this is the fifth partial sum of the sequence $\{\frac{1}{n+1}\}$

The arithmetic series

We start our study of summation with the $\boxed{\text{arithmetic sum.}}$

The sequence $\{a_1 + (n-1)d\}$ can be summed up as:

$$S_N = \underset{a_1}{a_1} + \underset{a_2}{(a_1+d)} + \underset{a_3}{(a_1+2d)} + \underset{a_4}{(a_1+3d)} + \cdots + \overbrace{[a_1+(N-1)d]}^{a_N}$$

- We can actually derive a formula for this sum by re writting the sum starting from the last term and decrementing each term by "d":

$$S_N = a_N + (a_N - d) + (a_N - 2d) + (a_N - 3d) + \cdots + a_N - (N-1)d$$

- Next we ADD these two sums together:

$$\boxed{S_N + S_N = (a_1 + a_N) + (a_1 + d + a_N - d) + (a_1 + 2d + a_N - 2d) + \cdots + (a_1 + (N-1)d + a_N - (N-1)d)}$$

- we see all terms with "d" cancell! resulting in:

$$2S_N = \overset{1}{(a_1+a_N)} + \overset{2}{(a_1+a_N)} + \overset{\cdots}{\cdots} + \overset{N}{(a_1+a_N)}$$

$$2S_N = N \cdot (a_1 + a_N)$$

$$\Rightarrow \boxed{S_N = N \cdot \left(\frac{a_1 + a_N}{2}\right)}$$

average the 1st & last terms then multiply by the number of terms present.

EX Use $S_N = \dfrac{N(a_1 + a_N)}{2}$ to sum the numbers 1 to 100

$$S_{100} = \overset{n=1}{1} + \overset{n=2}{2} + \overset{n=3}{3} + 4 + \cdots + 49 + 50 + 51 + \cdots + 97 + 98 + 99 + 100$$

100

100 …

100

$$= 49 \cdot 100 + 50 + 100$$

$$= 4900 + 150 = \boxed{5050}$$

→ formula

$$S_{100} = \frac{100 \cdot [1 + 100]}{2} = 50 \cdot 101$$

$$= 5000 + 50$$

$$= \boxed{5050}$$

EX Sum: $S = 1 + 2 + 3 + \cdots + 217 + 218$

$$S = \frac{218}{2}[1 + 218] = \boxed{23,871}$$

⊛ Need not be consequtive

$$a_n = \overset{41}{a} + (n-1)\overset{1}{d}$$
$$= 1 + (41-1)3$$

EX $S = 1 + 4 + 7 + 10 + 13 + \cdots + 121$

$n = 1 \quad 2 \xrightarrow{+3} 3 \xrightarrow{+3} 4 \xrightarrow{+3} 5 \quad \ldots \quad 41$

$$S_{41} = \frac{41 \cdot [1 + 121]}{2}$$

$$= \frac{41 \cdot 122}{2} = 41 \cdot 61 = \boxed{2501}$$

* arithmetic Sum (alt. formula)

• use the arithmetic general term formula...

$$S_N = \frac{N}{2}\left[a_1 + a_N\right] = \frac{N}{2}\left[2a_1 + (N-1)d\right]$$

but $a_N = a_1 + (N-1)d$ substitute into

to get $\boxed{S_N = N\left[a_1 + \left(\frac{N-1}{2}\right)d\right]}$

only needs • # terms to add (N)
• arithmetic difference (d)
• starting term

EX: What is the partial sum of the first 41 terms starting at $a_1 = 1$ with an arthemetic difference of 3?

$$S_{41} = \underset{1}{1} + \underset{2}{4} + \underset{3}{7} + \underset{4}{10} + \underset{\cdots}{\cdots} + \underset{41}{?}$$

$$= 41\left[1 + \frac{41-1}{2}(3)\right]$$

$$= 41\left[1 + 20 \cdot 3\right]$$

$$= 41\left[61\right]$$

$$= \boxed{2501}$$

* Geometric Series

- Recall the general term for a geometric sequence

$$\{a_n\} = \{a_1 \cdot r^{n-1}\} \quad , \quad \boxed{a_n = a_1 r^{n-1}}$$

- Sum these up now:

$$S_N = \sum_{n=1}^{N} a_1 r^{n-1}$$

- Expand out

$$\boxed{S_N = \overset{a_1}{a_1} + \overset{a_2}{a_1 r} + \overset{a_3}{a_1 r^2} + \cdots + \overset{a_N}{a_1 r^{N-1}}}$$

- mult. by "r"

$$r S_N = r a_1 + r a_1 r + r a_1 r^2 + \cdots + r a_1 r^{N-1}$$

$$\boxed{r S_N = r a_1 + a_1 r^2 + a_1 r^3 + \cdots a_1 r^N}$$

- Subtract these two forms...

$$S_N - r S_N = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots + a_1 r^{N-2} + \cancel{a_1 r^{N-1}}$$
$$- \cancel{r a_1} - \cancel{a_1 r^2} - \cancel{a_1 r^3} - \cdots - \cancel{a_1 r^{N-1}} - a_1 r^N$$

leaving only

$$S_N (1 - r) = a_1 - a_1 r^N$$

-OR- $$\boxed{S_N = a_1 \left(\frac{1 - r^N}{1 - r} \right)}$$

Formula for the partial sum for the geometric series.

$a_1 = 1^{st}$ term , $N = \#$ of terms

r = common ratio

Ex Sum these terms via the formula

$$\overset{n=1}{9} + \overset{2}{3} + \overset{3}{1} + \overset{4}{\tfrac{1}{3}} + \overset{5}{\tfrac{1}{9}} + \overset{6}{\tfrac{1}{27}} + \overset{7}{\tfrac{1}{81}}$$

$$9 \xrightarrow{*\frac{1}{3}} 3 \xrightarrow{*\frac{1}{3}} 1 \xrightarrow{*\frac{1}{3}} \tfrac{1}{3}$$

$a_1 = 9$, $r = ?\ \frac{1}{3}$, $N = 7$

$$\boxed{S_7 = 9\left(\frac{1-\left(\frac{1}{3}\right)^7}{\underbrace{1-\frac{1}{3}}_{2/3}}\right)} * \left(\frac{3^7}{3^7}\right)$$

$$\frac{1}{2/3} = \frac{3}{2}$$

$$= 9\left(\frac{3^7-1}{\frac{2}{3}\cdot 3^7}\right) = 9\cdot\left(\frac{3}{2}\right)\left(\frac{3^7-1}{3^7}\right)$$

$$= \frac{27}{2}\left[\frac{2187-1}{2187}\right] = \frac{27}{\cancel{2}}\left[\frac{\overset{1093}{\cancel{2186}}}{2187}\right]$$

$$= \boxed{\frac{29{,}511}{2187}} = \frac{9837}{729} = \frac{3279}{243} = \boxed{\frac{1093}{81}}$$

Ex Sum $\overset{n=1}{2} + \overset{2}{1.6} + \overset{3}{1.28} + \overset{4}{1.024} + \overset{5}{\dots}\ \overset{6}{}\ \overset{7}{}\ \overset{8}{}$

$$2 \xrightarrow{*0.8} 1.6 \xrightarrow{*0.8} 1.28 \xrightarrow{*0.8} 1.024$$

$$S_8 = 2\left[\frac{1-0.8^8}{1-0.8}\right] = 2\left[\frac{1-0.8^8}{0.2}\right]$$

$$= 10\left[1-0.8^8\right]$$

$$= 10\left[1-0.16777216\right]$$

$$= \boxed{8.3222784}$$

Ex: Find the sum of the 1st seven terms of a geom. seq. starting at 8 and having a common ratio of $\frac{1}{2}$.

$$S_N = a_1\left(\frac{1-r^N}{1-r}\right)$$

$$= 8\left(\frac{1-\left(\frac{1}{2}\right)^7}{1-\frac{1}{2}}\right)$$

$$= 8\left(\frac{1-\frac{1}{128}}{1/2}\right)$$

$$= 8\,\frac{\frac{128-1}{128}}{1/2}$$

$$= \frac{8\cdot 2}{128}\cdot 127$$

$$\boxed{8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}}$$

$$S_N = \frac{127}{8}$$

✱ ∞ geometric series

Observance:

$$1.3 \ \boxed{x^2} = 1.69$$
$$1.69 \ \boxed{x^2} = 2.8561$$
$$2.8561 \ \boxed{x^2} = 8.15730721$$

diverges...

get larger

So $\left(\left((1.3)^2\right)^2\right)^{2\cdots}$ ... ∞ big numbers

Next

$$0.8 \ \boxed{x^2} = 0.64$$
$$0.64 \ \boxed{x^2} = 0.4096$$
$$0.4096 \ \boxed{x^2} = 0.16777216$$

converges...

get smaller

So $\left(\left((0.8)^2\right)^2\right)^2 = (0.8)^8 \approx 0.168$

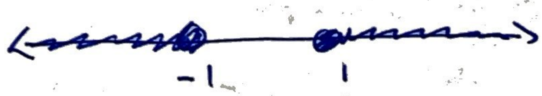
Conclusion

$$\boxed{\lim_{N\to\infty} \left(r^N\right) = \begin{cases} \infty & |r| \geq 1 \\ 0 & |r| < 1 \end{cases}}$$

• So the N^{th} partial sum of a geometric series converges if $|r| < 1$

$\boxed{-1 < r < 1}$

the series diverges if $|r| \geq 1$



$\boxed{(-\infty, -1] \cup [1, \infty)}$

That is $\boxed{S = a\left(\frac{1}{1-r}\right)}$ is the sum of $\boxed{S = \sum_{1}^{\infty} ar^{n-1}}$

Ex what is the sum of

$S_\infty = 1.3 + 1.69 + 2.861 + \cdots$

$= 1.3 + 1.3^2 + 1.3^3 + 1.3^4 + \cdots$

$= \infty$ since $r = 1.3$ is larger than 1

Ex what is the sum of

$S_\infty = 0.8 + 0.8^2 + 0.8^3 + \cdots$ $\quad r = 0.8 < 1$

$S_\infty = (0.8)\left(\frac{1}{1-0.8}\right)$

$= 0.8\left(\frac{1}{0.2}\right) = \frac{8}{2} = \boxed{4}$

Test...

$\left.\begin{array}{l} S_7 \approx 3.238 \\ S_{10} \approx 3.65 \end{array}\right\}$ converging slowly

Ex what does $\sum_{n=1}^{\infty} -\left[-\frac{1}{2}\right]^{n-1}$ converge to?

$= -\sum_{1}^{\infty}\left(-\frac{1}{2}\right)^{n-1}$

$r = -\frac{1}{2}$ & $|r| = \frac{1}{2} < 1$ converge

$= \underbrace{-\left[-\frac{1}{2}\right]^{1-1} \cdot \left(\frac{1}{1-(-1/2)}\right)}_{\text{formula} \leftarrow a_1\left(\frac{1}{1-r}\right)}$

$= -1 \cdot \frac{1}{3/2} = -\frac{2}{3}$ $\quad \boxed{S_\infty = -2/3}$

⊛ Application #1

- Recall an ∞ long decimal number w/o no repetition is an irrational number.

ex $\sqrt{2}$, $\sqrt{3}$, π, e

- Recall that any finite decimal number is a rational number {can be written as a fraction}

ex 2.456

$$2 + 0.456$$
$$= 2 + \frac{456}{1000}$$
$$= \frac{2000 + 456}{1000}$$
$$= \frac{2456}{1000} = \frac{1228}{500} = \frac{614}{250} = \boxed{\frac{307}{125}}$$

- Also any repetitive number can be be written as a fraction also:

ex $7.111111\cdots = 7.\overline{1}$

$$= 7 + 0.1 + 0.01 + 0.001 + \cdots$$
$$= 7 + \underbrace{(0.1)^1 + (0.1)^2}_{a_1 = 0.1,\ r = 0.1} + (0.1)^3 + \cdots$$
$$= 7 + \sum_{n=1}^{\infty} (0.1)^n$$
$$= 7 + 0.1\left(\frac{1}{1 - 0.1}\right)$$
$$= 7 + \frac{1}{10}\left(\frac{1}{1 - \frac{1}{10}}\right)$$
$$= 7 + \frac{1}{9}$$
$$= 63/9 + 1/9 = \boxed{\frac{64}{9}}$$

Ex Write $11.123123\cdots = 11.\overline{123}$

$$= 11 + 0.123 + 0.000123 + 0.000\,000\,123 + \cdots$$

$$= 11 + 123\left(\frac{1}{1000}\right) + 123\left(\frac{1}{1000}\right)^2 + 123\left(\frac{1}{1000}\right)^3 + \cdots$$

$$= 11 + 123\left[\underbrace{\frac{1}{1000}}_{a_1} + \left(\frac{1}{1000}\right)^2 + \left(\frac{1}{1000}\right)^3 + \cdots\right] \qquad r = 1/1000$$

$$= 11 + 123\left[\frac{1}{1000}\left(\frac{1}{1-\frac{1}{1000}}\right)\right]$$

$$= 11 + 123\left[\frac{1}{\cancel{1000}}\left(\frac{1}{\frac{1000-1}{\cancel{1000}}}\right)\right]$$

$$= 11 + \frac{123}{999}$$

$$= \boxed{\frac{11112}{999}}$$

Hack:

$$0.\overline{123} = \frac{123}{999}, \quad 0.\overline{4} = \frac{4}{9}$$

$$0.\overline{73} = \frac{73}{99}$$

So $\quad 0.\overline{abc} = \dfrac{abc}{999}, \quad 0.\overline{ab} = \dfrac{ab}{99}, \quad 0.\overline{a} = \dfrac{a}{9}$

Application

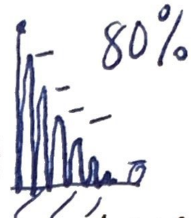
Arithmetic Sequences and Series (Cont.)

Example A pendulum swings 3'' shorter on each pass.

arithmetic

$\Delta = -3''$

BTW if each swing is 80% less



we have a geometric.

Q: If the first swing is 8', how far has the pedulum travelled in the 1st 12 swings?

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let a_{12} be the 12th swing, let $d = -3'' = -\frac{1}{4}'$

$$a_1 + a_2 + a_3 + \cdots + a_{11} + a_{12}$$

$$8' + \underline{7.75}' + \underline{7.5}' + \underline{7.25}' + \cdots$$

$$a_N = a_1 + (N-1)\cdot d = 8' + (12-1)\cdot\left(-\frac{1}{4}\right)$$

$$a_{12} = 8 - \frac{11}{4} = \frac{32}{4} - \frac{11}{4} = \frac{21}{4} = 5.25'$$

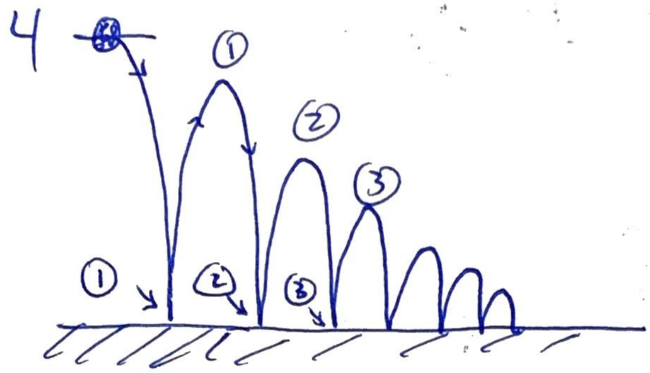
$$S_{12} = \frac{N(a_1 + a_{12})}{2} = \frac{12(8' + 5.25')}{2} = 6(13.25')$$

$$= \boxed{79.5'}$$

Application #3

A golf ball has a coefficient of restitution of 0.80, ie Each bounce is 80% the height as the previous bounce.

If a ball is dropped from 4 feet, what is the total distance, up and down after 12 bounces?



$$D_{TOT} = 4 + \overset{n=1}{0.8(4)\times 2} \quad (1)$$
$$+ \overset{n=2}{[0.8(4)]0.8 \times 2} \quad (2)$$
$$+ \overset{n=3}{[0.8(4)0.8]0.8 \times 2} \quad (3)$$
$$\vdots$$

- So $D = 4 + \sum_{n=1}^{12} (0.8)^n \times 8$

- Use formula $S_N = \sum_{1}^{N} ar^{n-1} = a\left(\frac{1-r^N}{1-r}\right)$

So need to get D formula in correct form:

$$D = 4 + \sum_{1}^{12} \underbrace{0.8}_{r}^{\,n-1} \underbrace{(0.8)\times 8}_{a}$$

$$= 4 + (0.8)(8)\left(\frac{1-0.8^{12}}{1-0.8}\right)$$

$$= 4 + \frac{0.64}{0.2}(1 - 0.068719477)$$

$$= 4 + 2.9800976$$

$$= \boxed{6.9801 \text{ feet}}$$ Traveled just befor the 13^{th} bounce occurs.

After ∞ many bounces

$$D = 4 + \frac{(0.8)(8)}{0.2}$$
$$= 4 + 32$$
$$= \boxed{36 \text{ ft}}$$

So a finite amount covered in an ∞ amount of time.