# Chapter 11 Sequences and Series

In Calc II we discore that we can express functions (transcendental) as $\infty$ series

ex $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$
for all values of $x$ between $-1$ and $1$

ex $e^x = 1 + x + \dfrac{x^2}{6} + \dfrac{x^3}{24} + \cdots \ \forall x$

## 11.1 Sequences

- A sequence is a set of numbers.
- A structured sequence has a pattern

ex Continue the sequence below:

$\{1, \frac{1}{2}, \frac{1}{3}, \underline{\ ?\ }\}$

- assign the "index" → $n=1$ $n=2$ $n=3$ $n=4$ $1/4$
- turn the index into the next term ⇒ $\{1, \frac{1}{2}, \frac{1}{3}, 1/4, \ldots, \boxed{1/n}, \ldots\}$

$a_1$ $a_2$ $a_3$ $a_4$ $a_n$

- establish the general term

$$\boxed{a_n = 1/n}$$

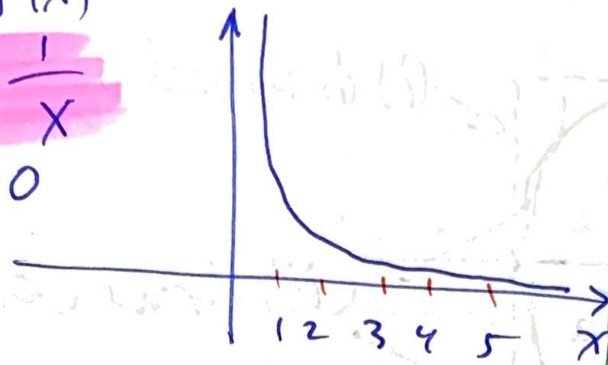
A sequence is also a function whose domain is the set of positive integers

Ex

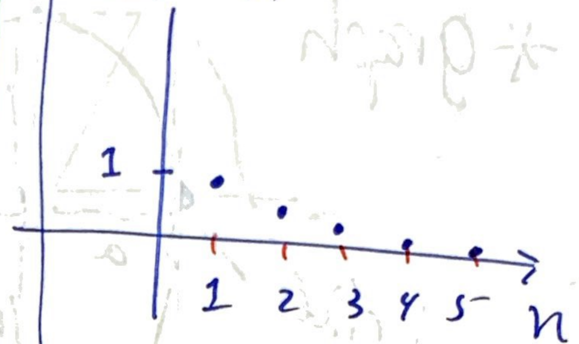
$y = f(x)$

$f = \frac{1}{x}$

$x > 0$




$a_n = \frac{1}{n}$



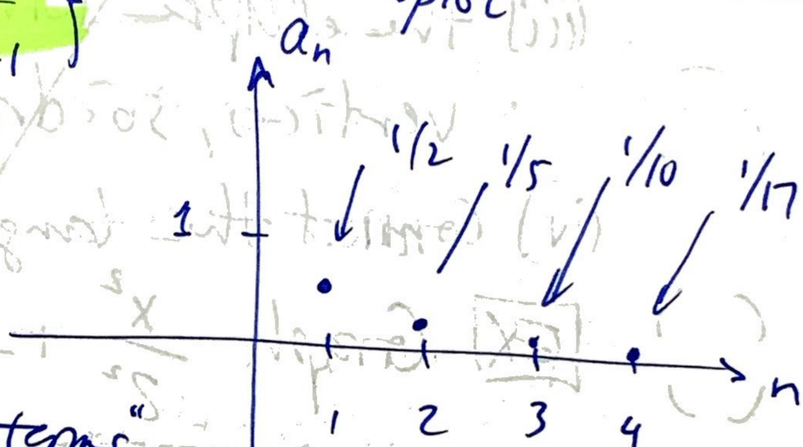

Discreet Function

Notation:

Ex

$\{a_n\} = \left\{ \frac{1}{n^2+1} \right\}$

• plot




• Expand the 1st "5-terms"

• value $= \{a_n\} = \left\{ \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26}, \ldots, \frac{1}{n^2+1}, \ldots \right\}$

• "pointer" $n = 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots n$

$\boxed{\text{index}}$

**Ex** Write out the 1st 5 terms of

$$\{a_n\} = \left\{\frac{2^n}{n}\right\}$$

$n=1$: $a_1 = \frac{2^1}{1} = 2$

$n=2$: $a_2 = \frac{2^2}{2} = 2$

$n=3$: $a_3 = 2^3/3 = 8/3$

$n=4$: $a_4 = 2^4/4 = 16/4 = 4$

$n=5$: $a_5 = 2^5/5 = 32/5$

$$\Rightarrow \{a_n\} = \left\{2, 2, \frac{8}{3}, 4, \frac{32}{5}, \dots\right\}$$

**Ex** Write the 1st 5 terms of the sequence whoes general term is $a_n = \frac{n^2}{2n+1}$

$a_1 = \frac{1^2}{2\cdot1+1} = \frac{1}{3}$

$a_2 = \frac{2^2}{2\cdot2+1} = \frac{4}{5}$

$a_3 = \frac{3^2}{2\cdot3+1} = \frac{9}{7}$

$a_4 = \frac{4^2}{2\cdot4+1} = \frac{16}{9}$

$a_5 = \frac{5^2}{2\cdot5+1} = \frac{25}{11}$

answer: $\left\{\frac{1}{3}, \frac{4}{5}, \frac{9}{7}, \frac{16}{9}, \frac{25}{11}, \dots, \frac{n^2}{2n+1}, \dots\right\}$

## ⊛ sequence aids

**factorial**:

EX

$0! = 1$

$1! = 1$

$2! = 1\cdot 2 = 2$

$3! = 1\cdot 2\cdot 3 = 6$

$\vdots$

$n! = 1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\cdot(n)$

OR backwards $n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$

Note that the factorial can be stopped before the end:

$$n! = (n)(n-1)! = n(n-1)(n-2)! = \ldots$$

EX $7! = 7\cdot 6\cdot 5!$

## ⊛ Recursive Sequences

These sequences utilize the previous value(s) of the sequence.

We need seed values, however:

EX

$$\boxed{a_n = n\cdot a_{n-1}}$$

let seed : $a_1 = 14$

then
$$\begin{cases} a_2 = 2\cdot a_1 = 2\cdot 14 = 28 \\ a_3 = 3\cdot a_2 = 3\cdot 28 = 84 \\ a_4 = 4\cdot a_3 = 4\cdot 84 = 336 \end{cases}$$

**Ex** a Fibonacci Sequence (there are many such)

$$\boxed{a_{n+2} = a_n + a_{n+1}}$$

two seeds: $a_1 = 2 \,,\, a_2 = -1$

then

$n = 1$: $a_{1+2} = a_1 + a_{1+1}$

$a_3 = a_1 + a_2$

$= (2) + (-1)$

$\boxed{a_3 = 1}$

$n = 2$: $a_{2+2} = a_2 + a_{2+1}$

$a_4 = a_2 + a_3$

$= (-1) + (1)$

$\boxed{a_4 = 0}$

$n = 3$: $a_{3+2} = a_3 + a_{3+1}$

$a_5 = a_3 + a_4$

$= 1 + 0$

$\boxed{a_5 = 1}$

$n = 4$: $a_{4+2} = a_4 + a_{4+1}$

$a_6 = a_4 + a_5$

$a_6 = 0 + 1$

$\boxed{a_6 = 1}$

$n = 5$: $a_{5+2} =$

$a_7 = a_5 + a_6$

$= (1) + (1)$

$\boxed{a_7 = 2}$

$n = 6$: $a_{6+2} =$

$a_8 = a_6 + a_7$

$= (1) + (2)$

$\boxed{a_8 = 3}$

$\vdots$

$\{a_n\} = \{2, -1, 1, 0, 1, 1, 2, 3 \dots\}$

**EX** Write the 1st 4 terms of the sequence whoes general term is $a_n = \dfrac{a_{n-1} + 2n}{a_{n-1} - 1}$ previous term

and whoes 1st term, $a_1$, is "$-4$"

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$a_1 = -4$

$a_2 = \dfrac{-4 + 2\cdot 2}{-4 - 1} = \dfrac{0}{-5} = 0$

$a_3 = \dfrac{0 + 2\cdot 3}{0 - 1} = -6$

$a_4 = \dfrac{-6 + 2\cdot 4}{-6 - 1} = \dfrac{2}{-7} = -\dfrac{2}{7}$

$a_5 = \dfrac{-\frac{2}{7} + 2\cdot 5}{-\frac{2}{7} - 1} = -\dfrac{68}{9}$

$$\boxed{\left\{-4, 0, -6, -\frac{2}{7}, -\frac{68}{9}, \ldots, \frac{a_{n-1} + 2n}{a_{n-1} - 1}, \ldots\right\}}$$

**EX** Write the general term for the sequence:

$\{4, 7, 12, 19, 28, 39\}$

$a_1$, $a_2 = a_1 + 3$, $a_3 = a_2 + 5$, $a_4 = a_3 + 7$

$$\begin{cases} a_1 = 4 \\ a_n = a_{n-1} + 2n - 1 \end{cases}$$

The latter examples were recursive relationships... they used the value(s) of the previous term(s)

EX What is the general term for

$$\{ 5, \underset{+6}{11}, \underset{+6}{17}, \underset{+6}{23}, \underset{+6}{29}, \dots \}$$

$n = 1 \quad 2 \quad 3 \quad 4 \quad 5 \leftarrow$ indecies

(i)

(ii) construct a few terms

$a_1 = 5$

$a_2 = 5 + 6$

$a_3 = 5 + 2 \cdot 6$

$a_4 = 5 + 3 \cdot 6$

(iii) State a formula

$$\boxed{a_n = 5 + n \cdot 6}$$

Not recursive !!

EX What is the general term for the sequence:

$$\left\{ \frac{1}{4}, \frac{1}{2}, 1, 2, \dots \right\}$$

multiply by $r = 2, 2, 2, \dots$

(i) seek the pattern

(ii) seek a formula

$$a_n = a_0 r^{n-1}$$

$$\boxed{a_n = \frac{1}{4} (2)^{n-1}}$$

$n = 1, 2, 3 \dots$

we could write $\frac{1}{4} = 2^{-2}$

(iii) State $a_n$

then $a_n = 2^{-2} 2^{n-1} = \boxed{2^{n-3}}$ $n = 1, 2, 3,$