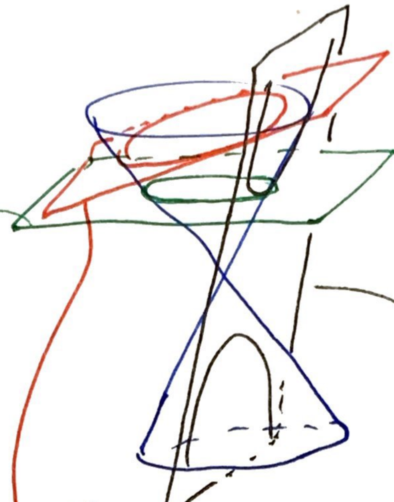
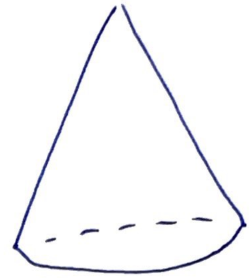
# Chapter 10 Conic Sections (Analytic Geom)



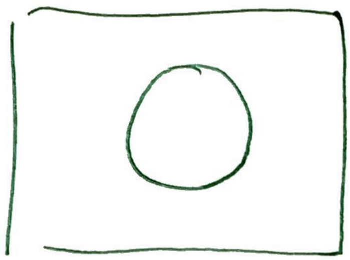
## Conic Sections



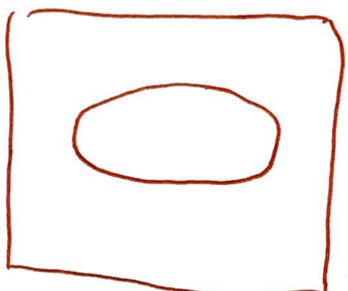
circ. cone



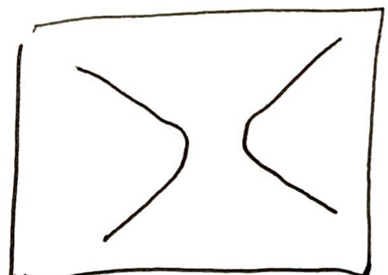
1/2 cone



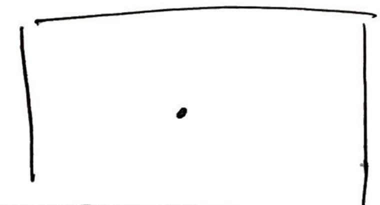
circle



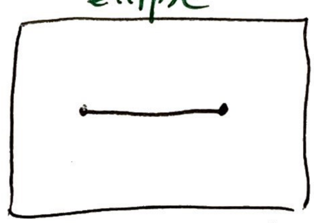
ellipse



Hyperbola

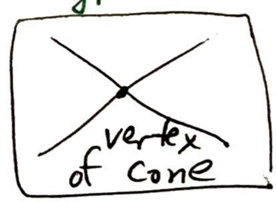


dot



?

line segment

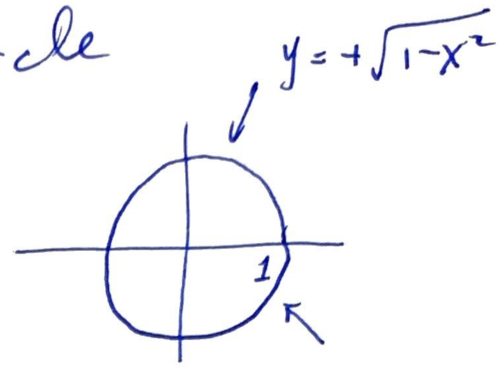



2 lines

# 10.1 Ellipses

Recall in parametrics that a circle

$$\left.\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \end{aligned}\right\} \text{ circle}$$




- use pythag. identity for trig:

$$\cos^2(t) + \sin^2(t) = 1$$

$$\boxed{x^2 + y^2 = 1}$$

Analytical Eqn of the circle.

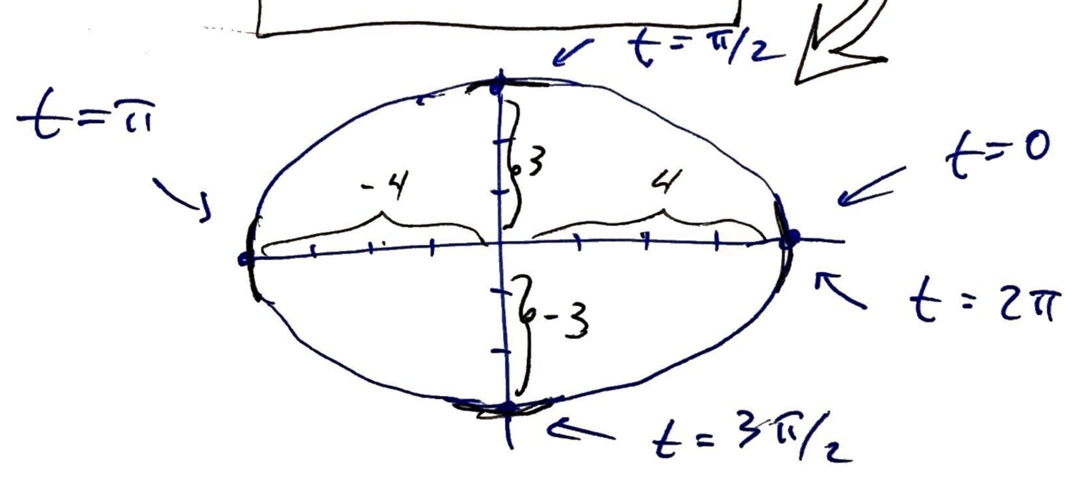
- we also discovered that if we changed the amplitude

$$\boxed{\left.\begin{aligned} x(t) &= 4\cos(t) \\ y(t) &= 3\sin(t) \end{aligned}\right\}} \text{ ellipse}$$

then $\cos^2(t) + \sin^2(t) = 1$ becomes

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\boxed{\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1}$$




avoid

## * std. form

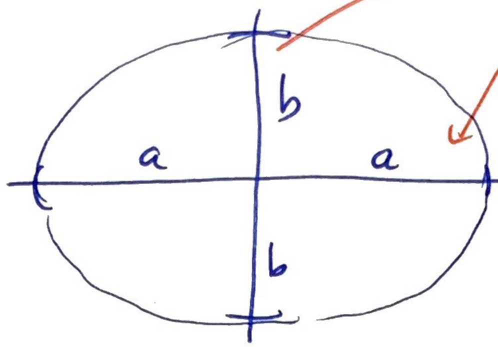
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$ *

if $b < a$

## * Graph

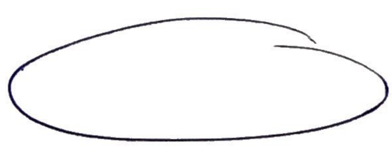
minor axis

major axis



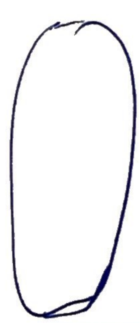

- the longer is called the major axis
  $\frac{1}{2}$ of it is called the semi-major axis
- the shorter is the minor-axis
- the extreme points are called the vertices

If $a > b$ we have a "lying down ellipse"
If $a < b$ we have a "standing up ellipse"



$a > b$

* 'a' always is the number under $x^2$ term
* 'b' under the $y^2$ term



$a < b$

**Ex** Graph $\dfrac{x^2}{5^2} + \dfrac{y^2}{2^2} = 1$

(i) std. form : $\dfrac{x^2}{5^2} + \dfrac{y^2}{2^2} = 1$

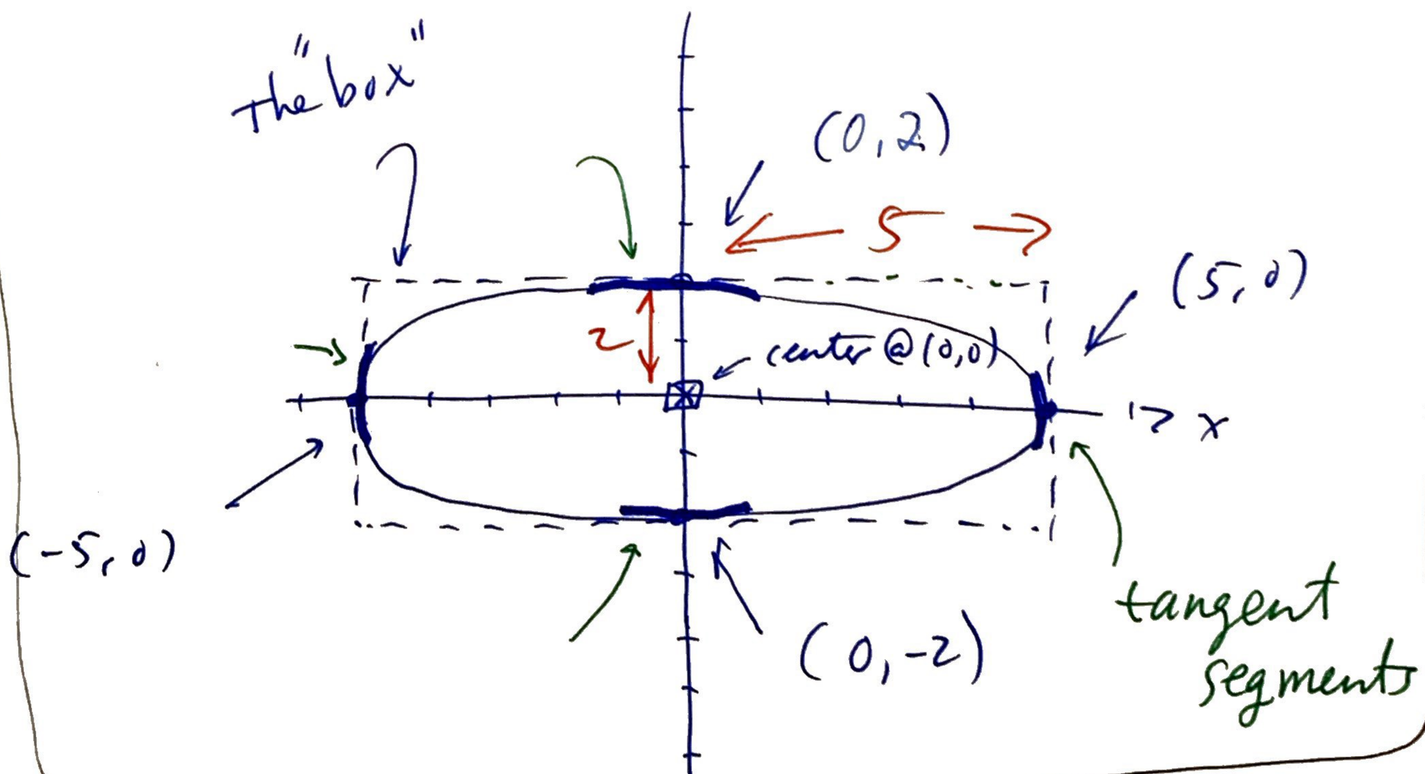
(ii) Identify $a$ & $b$ : $a = 5$, $b = 2$

{ I always put "a" under "x" and "b" under "y" }

(iii) plot these points; draw the "box" sides (- - - -)

(iv) draw the <u>tangent segments</u> ( $\perp$ to axii )

(v) connect the end of the segments.



Ex Graph $9x^2+4y^2-36=0$ [General form] 5

(i) Std. form: $\div 36$ $\boxed{\frac{9x^2}{36}+\frac{4y^2}{36}=1}$

$\div$ Top & bottom of $\frac{9}{36}$ by 9

$\div$ Top & bottom of $\frac{4}{36}$ by 4

$$\Rightarrow \frac{9x^2/9}{36/9}+\frac{4y^2/4}{36/4}=1$$

$$\frac{x^2}{\left(36/9\right)}+\frac{y^2}{\left(36/4\right)}=1$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

show as squares...

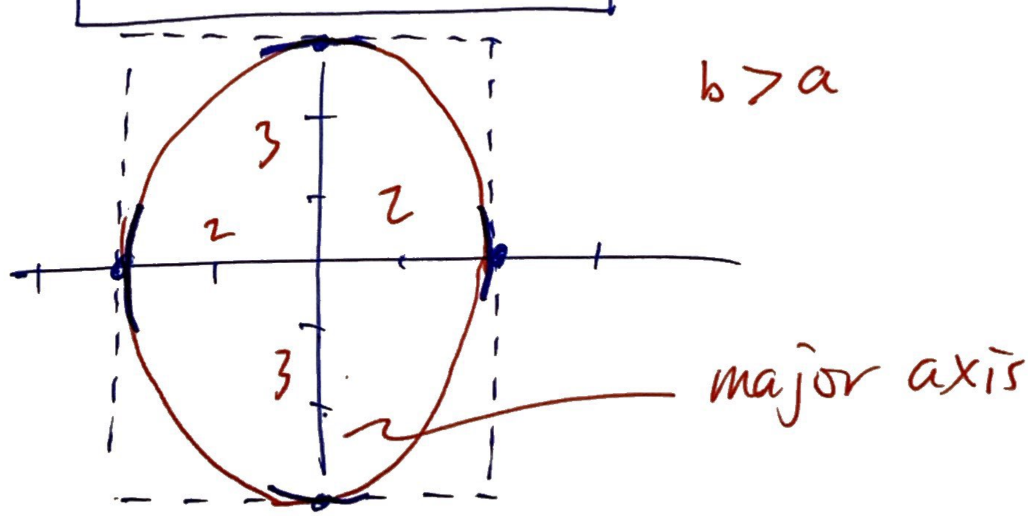
$$\boxed{\frac{x^2}{2^2}+\frac{y^2}{3^2}=1}$$

Std Form

(ii) Box

(iii) tangent segments

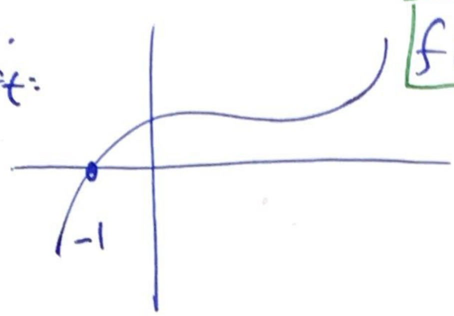
(iv) Connect

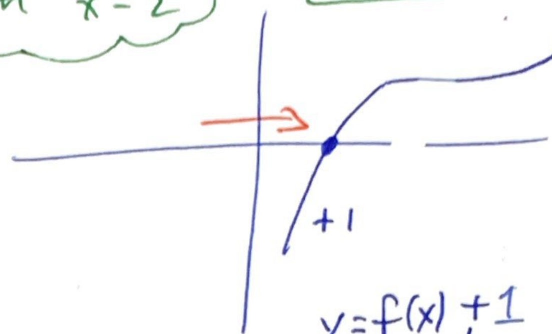


## ⊛ off-origin centers

• horiz. shift:

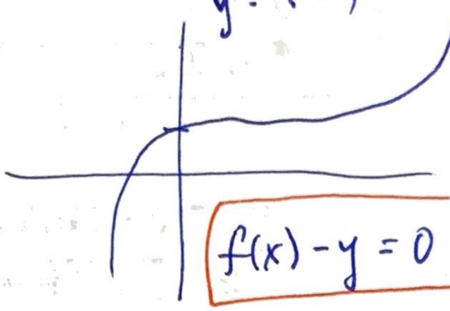
$\boxed{f(x)}$ → replace $x$ with $x-2$ → $\boxed{f(x-2)}$



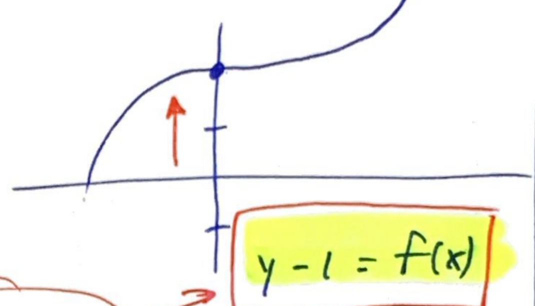




• vertical shift

$y = f(x)$



$y = f(x) + 1$



$\boxed{f(x) - y = 0}$ → replace $y$ with $y-1$ → $\boxed{y - 1 = f(x)}$

So $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is centred @ $(x,y) = (0,0)$

but $\boxed{\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1}$ is centred @ $(h,k)$

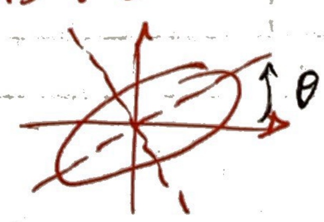
full std. Form.

Conic Section

• Likewise $\boxed{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$

is the full General form of a conic section.

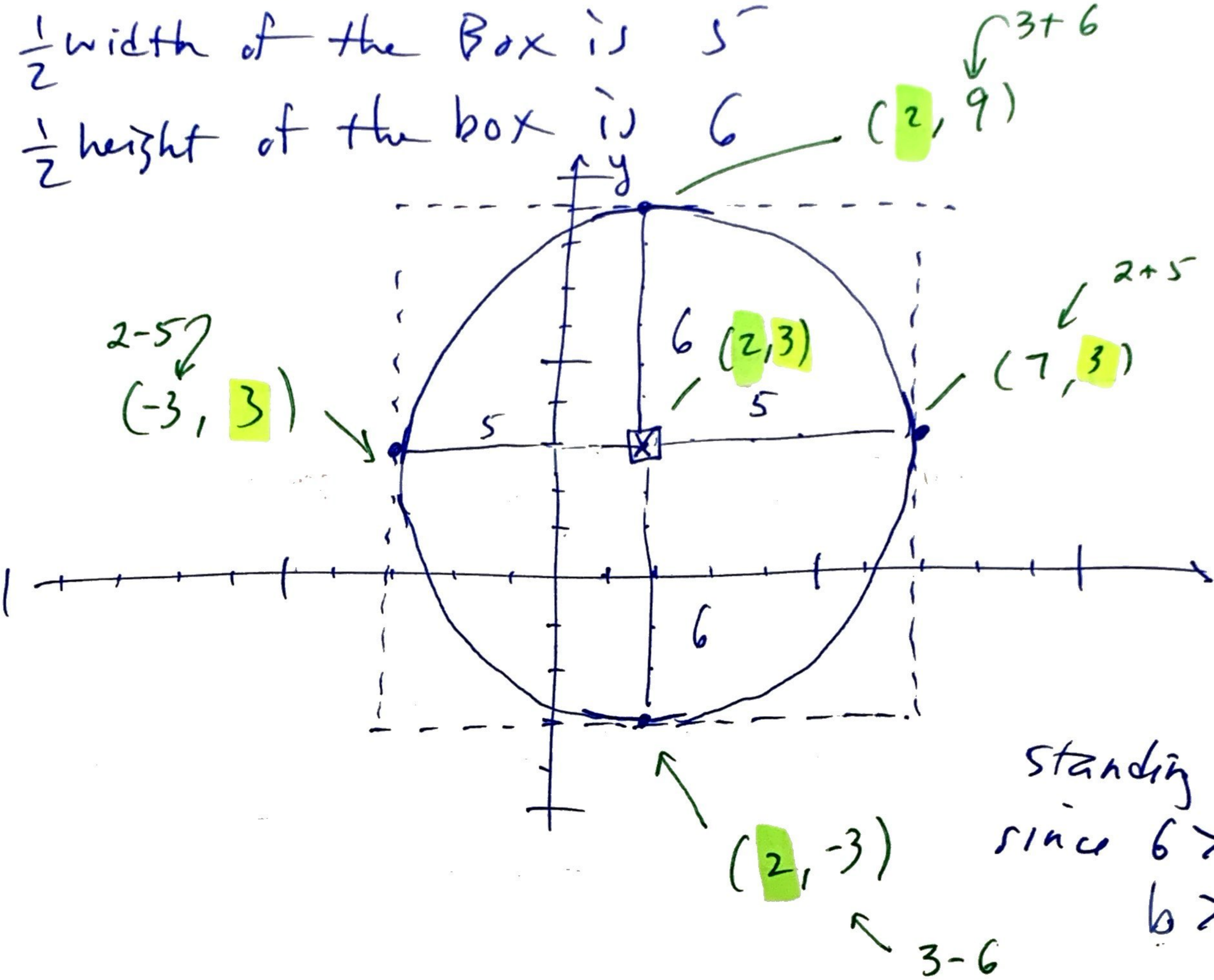
NOTE: if $B \neq 0$ then the conic section is rotated




$$\theta = \frac{1}{2}\cot^{-1}\left(\frac{A-C}{B}\right)$$

$\boxed{Ex}$ Sketch $\dfrac{(x-2)^2}{5^2} + \dfrac{(y-3)^2}{6^2} = 1$

- center is @ $(2,3)$
- $\frac{1}{2}$ width of the Box is 5
- $\frac{1}{2}$ height of the box is 6




standing ellipse since $6 > 5$

$b > a$

**Ex**

Write $9x^2 + 72x + 16y^2 + 16y + 4 = 0$ in standard form.

ellipse since $(+)x^2$ & $(+)y^2$ and $9 \neq 16$.

---

complete the square (calculus way):

→ factor:

$$9(x^2 + 8x) + 16(y^2 + y) = -4$$

→ magic zeros:

$$9\left(\underbrace{x^2 + 8x + \left(\tfrac{8}{2}\right)^2}_{} - \left(\tfrac{8}{2}\right)^2\right) + 16\left(\underbrace{y^2 + 1y + \left(\tfrac{1}{2}\right)^2}_{} - \left(\tfrac{1}{2}\right)^2\right) = -4$$

→ complete the square

$$9\left[(x+8)^2 - \left(\tfrac{8}{2}\right)^2\right] + 16\left[(y+1)^2 - \left(\tfrac{1}{2}\right)^2\right] = -4$$

→ clean up

$$9(x+8)^2 - 9 \cdot 16 + 16 \cdot (y+1)^2 - 16 \cdot \tfrac{1}{4} = -4$$

$$9(x+8)^2 + 16(y+1)^2 = -4 + 9 \cdot 16 + 16 \cdot \tfrac{1}{4}$$

$$9(x+8)^2 + 16(y+1)^2 = \cancel{-4} + 144 + \cancel{4}$$

$\div 144$

$$\frac{9}{144}(x+8)^2 + \frac{16}{144}(y+1)^2 = 1$$

$$\frac{(x+8)^2}{\left(\frac{144}{9}\right)} + \frac{(y+1)^2}{\left(\frac{144}{16}\right)} = 1$$

$$\boxed{\frac{(x+8)^2}{4^2} + \frac{(y+1)^2}{3^2} = 1}$$

# ⊗ Focii

Another way to describe an ellipse is to use the "locus of points" method.

**Def:** An ellipse is the locus of points, (all points) that satisfy the following

$$l_1 + l_2 = \text{constant}$$

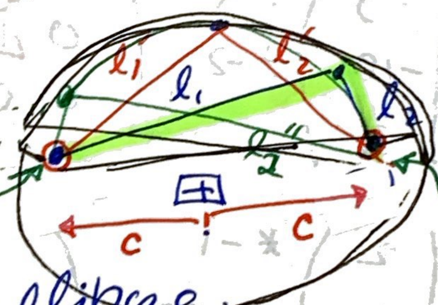
where $l_1$ is the distance from a fixed and $l_2$ is the distance from a different fixed point.

$l_1 + l_2 = d$

$l_1' + l_2' = d$

$l_1'' + l_2'' = d$

Focus #1




Focus #2

⊗ Def. of "c":

- Laying down ellipses: $a > b$

$$\boxed{a^2 - b^2 = c^2}$$ where "c" is the $\frac{1}{2}$ distance between focii

- Standing Ellipse $a < b$

$$\boxed{b^2 - a^2 = c^2}$$

$e = c / \max(a,b)$ eccentricity.

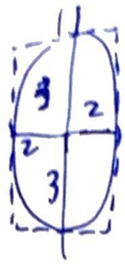
$$\Rightarrow \boxed{[\max(a,b)]^2 - [\min(a,b)]^2 = c^2}$$

Ex Find the focii of

$$\frac{x^2}{2^2}+\frac{y^2}{3^2}=1 \qquad \boxed{b^2-a^2=c^2} \text{ if } b>a$$

$$\boxed{a^2-b^2=c^2} \text{ if } a>b$$

- sketch
  - → $a=2$
  - → $b=3$




- stand-up ellipse so use $3^2-2^2=c^2$

$$3^2-2^2=c^2$$

$$9-4=c^2$$

$$5=c^2$$

$$c=\pm\sqrt{5}$$

$$c\approx\pm\,2.3$$

Focii are on the major axis

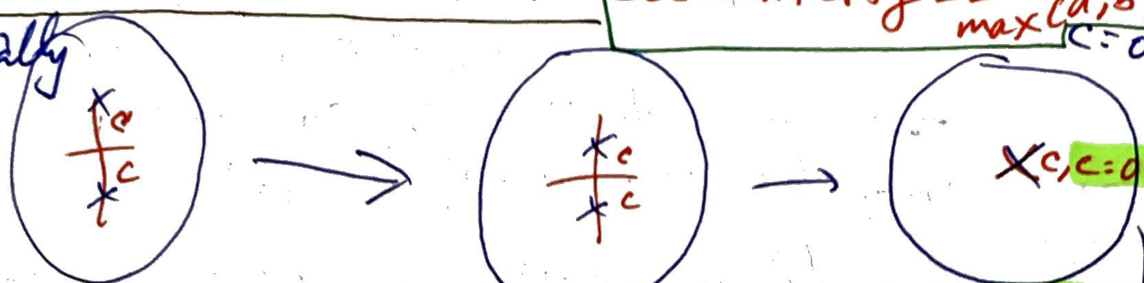

$(0,+\sqrt{5})$

$(0,-\sqrt{5})$

$$\text{eccentrizity}=\frac{c}{\max(a,b)}$$

(*) geometrically

$b>a$

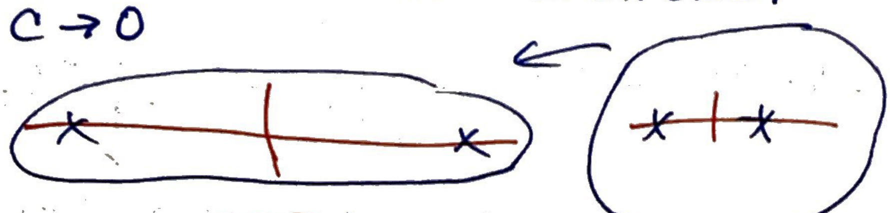



circle

$e=0$

- as an ellipse becomes more circular then $c\to 0$

$a>b$



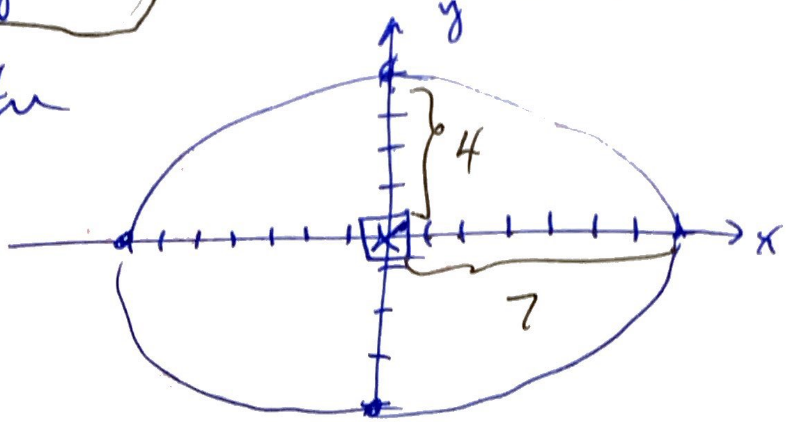
$e>>1$

# * C.S.I. problems



Work from the details to reverse engineer the eqn.

**EX** given the graph




- What is the eqn?

(i) Start by writing the Std. Form:

$$\text{Form} \quad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

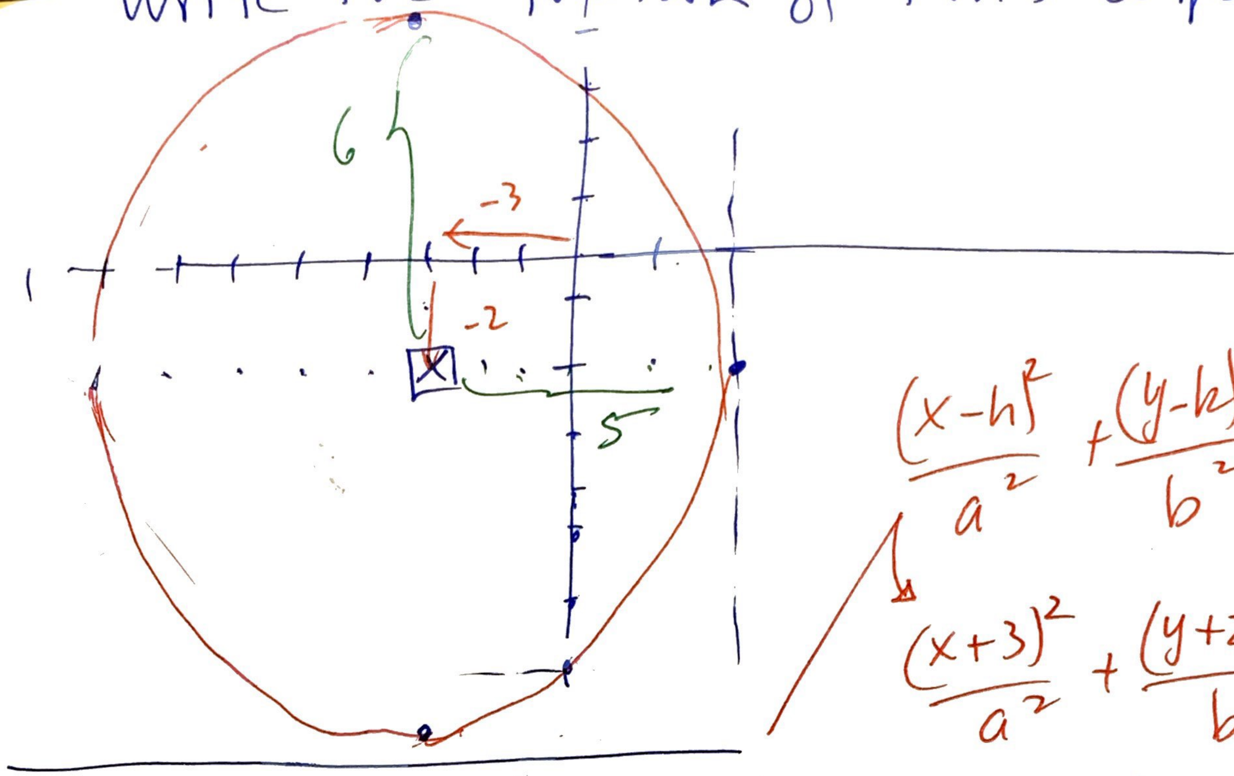
(ii) Start to populate h, k, a, b

- $(h,k) = (0,0) \longrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- $(a,b) = (7,4)$

(iii) state final answer:

$$\boxed{\frac{x^2}{7^2} + \frac{y^2}{4^2} = 1}$$

**Ex** Write the equation of this ellipse.




$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

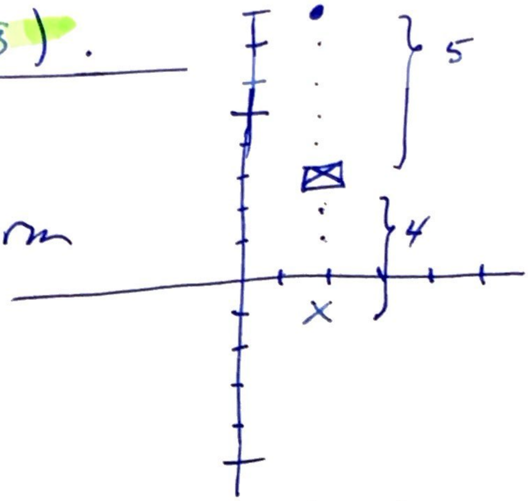
$$\frac{(x+3)^2}{a^2}+\frac{(y+2)^2}{b^2}=1$$

- center $(h,k)=(-3,-2)$
- $(a,b)=(5,6)$
- final form:

$$\boxed{\frac{(x+3)^2}{5^2}+\frac{(y+2)^2}{6^2}=1}$$

Ex 9

Find the equation of an ellipse whose center is at $(2,3)$ and a focus is at $(2,-1)$ and a vertex at $(2,8)$.




(i) plot the given data

(ii) Populate the std. Form

- $(h,k) = (2,3)$
- $b = 8-3 = 5$
- $c = 3-(-1) = 4$

use $b^2 - a^2 = c^2$ since this is stand-up ellipse {we know this b/c the focus is on the major axis}

$$5^2 - a^2 = 4^2$$
$$-a^2 = 4^2 - 5^2$$
$$a^2 = 25-16$$
$$a^2 = 9$$
$$a = 3$$

(iii) Final answer

$$\boxed{\frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{5^2} = 1}$$

c/w mod #4 cont.

$|a^2 - b^2| = c^2$

ellipses

$\boxed{\#9}$ Sketch $\frac{(x-2)^2}{81} + \frac{(y+1)^2}{16} = 1$

label center, all vertices, and foci.

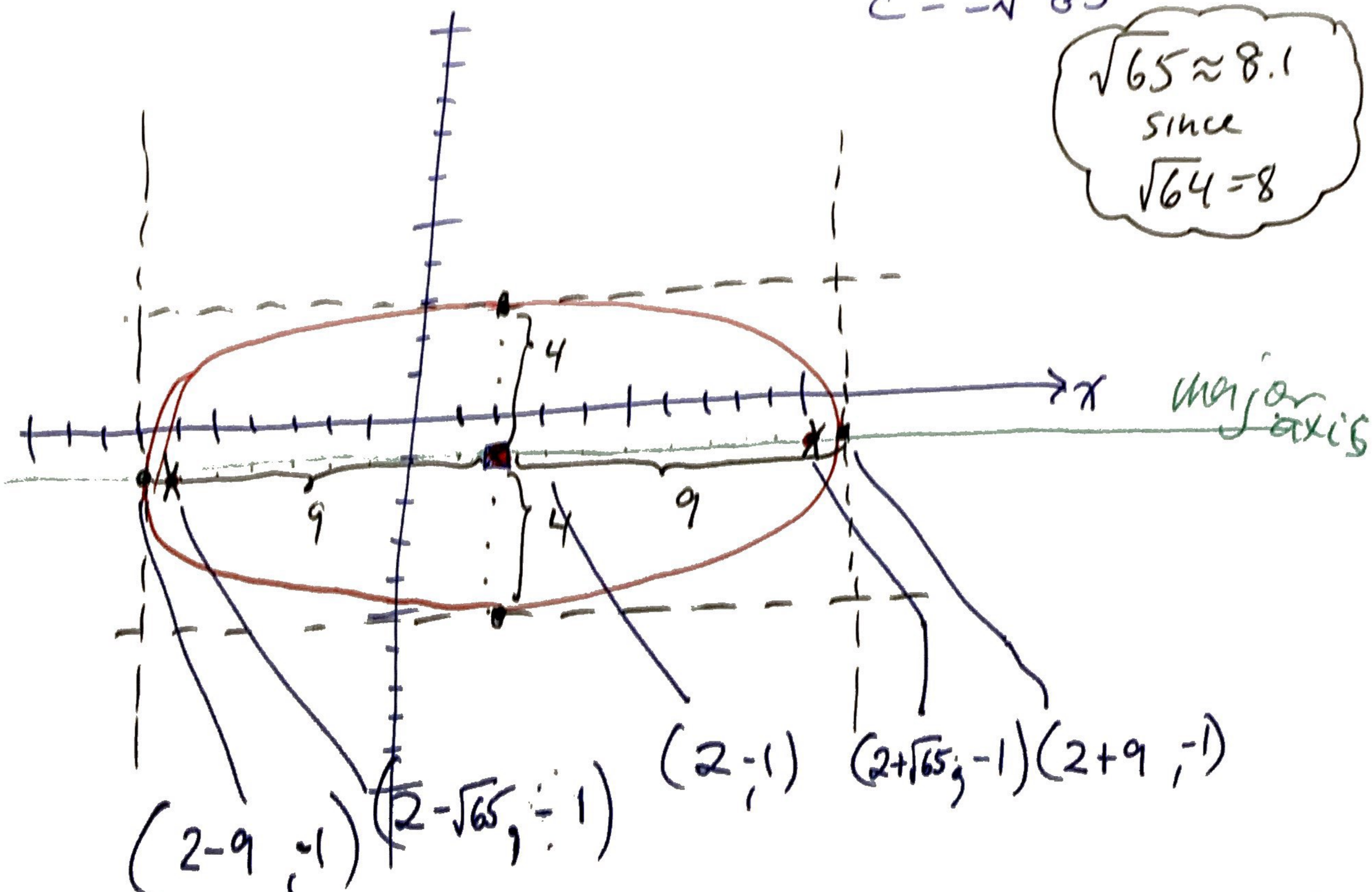
$$\frac{(x-2)^2}{9^2} + \frac{(y+1)^2}{4^2} = 1$$

- $a = 9$, $b = 4$
- center @ $(2,-1)$
- foci @ $\pm\sqrt{9^2 - 4^2}$

$= \pm\sqrt{81-16}$

$c = \pm\sqrt{65}$

$\sqrt{65} \approx 8.1$ since $\sqrt{64} = 8$




$(-7,-1)$, $(2-\sqrt{65},-1)$, $(2,-1)$, $(2+\sqrt{65},-1)$, $(11,-1)$