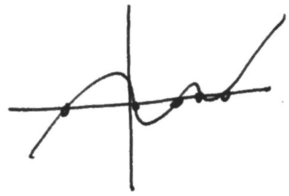
ZEROS OF POLYNOMIALS



Why: Many phenomena are modelled with poly's.

Objective: $f(x) = 0$ @ $x = a, b, c, \ldots$

Q: How do we find these zeros (roots, x-int.) aka.

How:
We use **theorems** that estimate various facts about the quantity, locations & candidate values of these zeros.

We use **tools** to help us evaluate f @ various x's.

Tools First

* Recall $\frac{13}{2} = 6r1$ or $\frac{13}{2} = 6 + \frac{1}{2}$

likewise $\frac{f(x)}{q(x)} = g(x), r(x)$ or $= g(x) + \frac{r(x)}{q(x)}$

clear the fractions: $\boxed{f(x) = q(x)g(x) + r(x)}$

{we use long-division to obtain the above g & r}

* Now if $q(x)$ is a linear binomial, $(x-d)$, then $r(x) =$ constant : $\boxed{f(x) = (x-d)\cdot g(x) + r}$

{we use synthetic division to obtain the above g & r}

* Note that @ $x = d$: $f(d) = (d-d)\cdot g(x) + r$

or $\boxed{f(d) = r}$ Remainder Thm

* Therefore if $r = 0$ then $f(d) = 0$ so "d" is a zero!

We have three ways to obtain "r"

a. plug & chug $f(d) = d^3 + 5d^2 + \cdots$ etc.

b. long divide $x-d$ into $f(x)$

c. synthetic division of $d \rfloor a_n\ a_{n-1} \cdots a_0$

* Assuming we have located one zero, $x = d$, note that

$$f(x) = (x-d)\,g(x) + 0$$

$$\boxed{f(x) = (x-d)\,g(x)}$$

Factor Thm

we have partially factored $f(x)$.

* The next zero will be found from $g(x) = 0$, called the reduced Polynom.

Let $x = e$ be one such zero of $g(x)$ then

$g(x) = (x-e)\,h(x)$ results

* Continuing this process we will eventually factor $f(x)$ into a product of linear binomials

$$\boxed{f(x) = (x-d)(x-e)(x-f)\cdots(x-p)}$$

Quantity

* So where do we start? How about **Quantity**

Q: How many zeros does a polynomial have?

A: $\boxed{\text{If the degree of the poly is "}n\text{", } \exists \text{ "}n\text{" zeros.}}$

{ some may be complex: $a+bi$, Which BTW come in pairs: $a \pm bi$ }

Q: How many could by positive numbers?

A: Use Des Cartes rule of signs on $f(x)$

Q: How many could by negative numbers?

A: Use Des Cartes rule of signs on $f(-x)$

Thm: Let f denote a poly. in std form:

$\oplus$: The # of $\oplus$ zeros of f = # sign variations or else an even number less than that #.

$\ominus$: The # of $\ominus$ zeros of f = # of sign variations of $f(-x)$, or else an even number less than that

Candidates

* What values do we start with?

Rational zeros Thm:

let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0$

If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p is a factor of a_0, and q is a factor of a_n

* Basically we can then start by writing out all rational #'s formed by the Factors of a_0 ÷ by Factors of a_n:

$$\pm \frac{\text{Factors } a_0}{\text{Factors } a_n}$$

Bounds

* We need not examine numbers larger than "b" if the synthetic division of b into f yeilds a bottom line consisting of all pos. #'s

* We need not examine numbers lower than "l" if the synthetic division of l into f yeilds a bottom line consisting of alternating #'s.

ex:

$$\begin{array}{r|rrrr} 2 & 3 & 11 & -6 & -8 \\ & & 6 & 34 & 56 \\ \hline & 3 & 17 & 28 & 48 \end{array}$$

← all (+) therefore no zero is larger than 2.

ex:

$$\begin{array}{r|rrrr} -8 & 3 & 11 & -6 & -8 \\ & & -24 & 104 & -784 \\ \hline & 3 & -13 & 98 & \cancel{776} \\ & & & & -792 \end{array}$$

← altenating signs, ∴ no zero is less than -8

"Desperately Seeking Zeros"

A summary of zero location tools:

1. What degree is the polynomial. Thats how many zeros to expect. {some may be complex conjugates}

p316 #40 ex: $P(x) = 3x^3 + 11x^2 - 6x - 8$ degree $= 3$

$\boxed{1, -4, -\frac{2}{3}}$

2. Number of "+" zeros: (Descartes Rule of Signs)

ex: $P(+x) = 3x^3 + 11x^2 - 6x - 8$

1 positive root

3. Num of "−" zeros:

ex: $P(-x) = 3(-x)^3 + 11(-x)^2 - 6(-x) - 8$

$= -3x^3 + 11x + 6x - 8$

1 2

expect 2 or $\emptyset$

4. Rational zeros: List possible zeros

Factors of a_0

$\frac{p}{q} = \pm \frac{1, 2, 4, 8}{1, 3} = \frac{1}{1}, \frac{2}{1}, \frac{4}{1}, \frac{8}{1}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}$

Factors of a_n

$= \left(\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3} \pm \frac{8}{3}\right)$

$-8, -4, -\frac{8}{3}, -2, -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, 2, \frac{8}{3}, 4, 8$

$\boxed{5}$ Upper Bound: ~~Lower Bound~~ For a_n positive: Seek lowest of the upper numbers that yields an all "+" Bottom Line

ex $P = 3x^3 + 11x^2 - 6x - 8$

$$\text{* try } 2 \,\big|\; \begin{array}{cccc} 3 & 11 & -6 & -8 \\ & 6 & 34 & 56 \\ \hline 3 & 17 & 28 & 48 \end{array}$$

all + No zero is greater than 2

$\boxed{6}$ Lower Bound

$$\text{* try } -2 \,\big|\; \begin{array}{cccc} 3 & 11 & -6 & -8 \\ & -6 & -10 & 16 \\ \hline 3 & 5 & -16 & 8 \end{array}$$

not alternating #'s so $\exists$ a zero < -2

$$\text{* try } -8 \,\big|\; \begin{array}{cccc} 3 & 11 & -6 & -8 \\ & -24 & 104 & -784 \\ \hline 3 & -13 & 98 & -7\cancel{76} \end{array}$$

792

alt. # so -8 is a lower bound

So zero's $\in (-8, 2) = -4, -\frac{8}{3}, -2, -\frac{4}{3}, -1, -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}$

$\boxed{7}$ Locate 1st Root, then use reduced poly:

$$\text{* try } 1 \,\big|\; \begin{array}{cccc} 3 & 11 & -6 & -8 \\ & 3 & 14 & 8 \\ \hline 3 & 14 & 8 & 0 \end{array}$$

easy #'s

A ZERO!

So $P = 3x^3 + 11x^2 - 6x - 8 = (3x^2 + 14x + 8)(x - 1)$

* Zero's of the reduced poly:

$3x^2 + 14x + 8$

$\begin{matrix} 4,2 \\ 8,1 \end{matrix}$

Try Trial & error: $\overbrace{(3x \quad \underbrace{4)(x}_{\cancel{8x}} \quad 2)}^{6x}$

$(3x+2)(x+4)$ ✓

$x = -4, -\frac{2}{3}$

So $P(x) = 3x^3 + 11x^2 - 6x - 8 = \underline{(x+4)(3x+2)(x-1)}$

Zero's are $\boxed{1, -\frac{2}{3}, -4}$ Done!!